Chapter 1

PRELIMINARIES

OVERVIEW This chapter reviews the basic ideas you need to start calculus. The topics include the real number system, Cartesian coordinates in the plane, straight lines, parabolas, circles, functions, and trigonometry. We also discuss the use of graphing calculators and computer graphing software.

1.1 Real Numbers and the Real Line

This section reviews real numbers, inequalities, intervals, and absolute values.

Real Numbers

Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

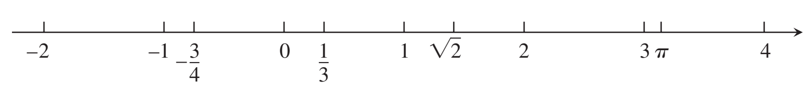
$$-\frac{3}{4} = -0.75000\ldots$$

$$\frac{1}{3} = 0.33333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

The dots $\ldots$ in each case indicate that the sequence of decimal digits goes on forever. Every conceivable decimal expansion represents a real number, although some numbers have two representations. For instance, the infinite decimals $.999\ldots$ and $1.000\ldots$ represent the same real number 1. A similar statement holds for any number with an infinite tail of 9's.

The real numbers can be represented geometrically as points on a number line called the **real line**.



The symbol $\mathbb{R}$ denotes either the real number system or, equivalently, the real line.

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The **algebraic properties** say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by* 0.

The **order properties** of real numbers are given in Appendix 4. The following useful rules can be derived from them, where the symbol $\Rightarrow$ means "implies."

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \dfrac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

Notice the rules for multiplying an inequality by a number. Multiplying by a positive number preserves the inequality; multiplying by a negative number reverses the inequality. Also, reciprocation reverses the inequality for numbers of the same sign. For example, $2 < 5$ but $-2 > -5$ and $1/2 > 1/5$.

The **completeness property** of the real number system is deeper and harder to define precisely. However, the property is essential to the idea of a limit (Chapter 2). Roughly speaking, it says that there are enough real numbers to "complete" the real number line, in the sense that there are no "holes" or "gaps" in it. Many theorems of calculus would fail if the real number system were not complete. The topic is best saved for a more advanced course, but Appendix 4 hints about what is involved and how the real numbers are constructed.

We distinguish three special subsets of real numbers.

1. The **natural numbers**, namely $1, 2, 3, 4, \ldots$
2. The **integers**, namely $0, \pm 1, \pm 2, \pm 3, \ldots$
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction m/n, where m and n are integers and $n \neq 0$. Examples are

$$\frac{1}{3}, \quad -\frac{4}{9} = \frac{-4}{9} = \frac{4}{-9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

The rational numbers are precisely the real numbers with decimal expansions that are either

(a) terminating (ending in an infinite string of zeros), for example,

$$\frac{3}{4} = 0.75000\ldots = 0.75 \qquad \text{or}$$

(b) eventually repeating (ending with a block of digits that repeats over and over), for example

$$\frac{23}{11} = 2.090909\ldots = 2.\overline{09}$$

The bar indicates the block of repeating digits.

A terminating decimal expansion is a special type of repeating decimal since the ending zeros repeat.

The set of rational numbers has all the algebraic and order properties of the real numbers but lacks the completeness property. For example, there is no rational number whose square is 2; there is a "hole" in the rational line where $\sqrt{2}$ should be.

Real numbers that are not rational are called **irrational numbers**. They are characterized by having nonterminating and nonrepeating decimal expansions. Examples are π, $\sqrt{2}$, $\sqrt[3]{5}$, and $\log_{10} 3$. Since every decimal expansion represents a real number, it should be clear that there are infinitely many irrational numbers. Both rational and irrational numbers are found arbitrarily close to any point on the real line.

Set notation is very useful for specifying a particular subset of real numbers. A **set** is a collection of objects, and these objects are the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $a \notin S$ means that a is not an element of S. If S and T are sets, then $S \cup T$ is their **union** and consists of all elements belonging either to S or T (or to both S and T). The **intersection** $S \cap T$ consists of all elements belonging to both S and T. The **empty set** $\varnothing$ is the set that contains no elements. For example, the intersection of the rational numbers and the irrational numbers is the empty set.

Some sets can be described by *listing* their elements in braces. For instance, the set A consisting of the natural numbers (or positive integers) less than 6 can be expressed as

$$A = \{1, 2, 3, 4, 5\}.$$

The entire set of integers is written as

$$\{0, \pm 1, \pm 2, \pm 3, \dots\}.$$

Another way to describe a set is to enclose in braces a rule that generates all the elements of the set. For instance, the set

$$A = \{x \mid x \text{ is an integer and } 0 < x < 6\}$$

is the set of positive integers less than 6.

Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements. For example, the set of all real numbers x such that $x > 6$ is an interval, as is the set of all x such that $-2 \le x \le 5$. The set of all nonzero real numbers is not an interval; since 0 is absent, the set fails to contain every real number between -1 and 1 (for example).

Geometrically, intervals correspond to rays and line segments on the real line, along with the real line itself. Intervals of numbers corresponding to line segments are **finite intervals**; intervals corresponding to rays and the real line are **infinite intervals**.

A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint but not the other, and **open** if it contains neither endpoint. The endpoints are also called **boundary points**; they make up the interval's **boundary**. The remaining points of the interval are **interior points** and together comprise the interval's **interior**. Infinite intervals are closed if they contain a finite endpoint, and open otherwise. The entire real line $\mathbb{R}$ is an infinite interval that is both open and closed.

Solving Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

EXAMPLE 1 Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\dfrac{x}{3} < 2x + 1$ **(c)** $\dfrac{6}{x-1} \ge 5$

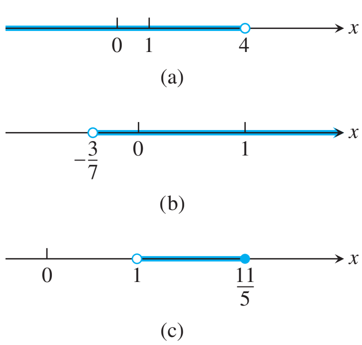


FIGURE 1.1 Solution sets for the inequalities in Example 1.

Solution

(a)

$$
\begin{aligned}
2x - 1 &< x + 3 \\
2x &< x + 4 && \text{Add 1 to both sides.} \\
x &< 4 && \text{Subtract } x \text{ from both sides.}
\end{aligned}
$$

The solution set is the open interval $(-\infty, 4)$ (Figure 1.1a).

(b)

$$
\begin{aligned}
-\frac{x}{3} &< 2x + 1 \\
-x &< 6x + 3 && \text{Multiply both sides by 3.} \\
0 &< 7x + 3 && \text{Add } x \text{ to both sides.} \\
-3 &< 7x && \text{Subtract 3 from both sides.} \\
-\frac{3}{7} &< x && \text{Divide by 7.}
\end{aligned}
$$

The solution set is the open interval $(-3/7, \infty)$ (Figure 1.1b).

(c) The inequality $6/(x-1) \geq 5$ can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative. Therefore, $(x-1)$ is positive and the inequality will be preserved if we multiply both sides by $(x-1)$, and we have

$$\begin{aligned} \frac{6}{x-1} &\geq 5 & \\ 6 &\geq 5x - 5 & &\text{Multiply both sides by } (x-1). \\ 11 &\geq 5x & &\text{Add 5 to both sides.} \\ \frac{11}{5} &\geq x. & &\text{Or } x \leq \frac{11}{5}. \end{aligned}$$

The solution set is the half-open interval $(1, 11/5]$ (Figure 1.1c). ■

Absolute Value

The **absolute value** of a number x, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

EXAMPLE 2 Finding Absolute Values

$$|3| = 3, \quad |0| = 0, \quad |-5| = -(-5) = 5, \quad |-|a|| = |a|$$ ■

Geometrically, the absolute value of x is the distance from x to 0 on the real number line. Since distances are always positive or 0, we see that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$. Also,

$$|x - y| = \text{the distance between } x \text{ and } y$$

on the real line (Figure 1.2).

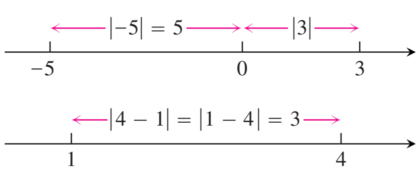


FIGURE 1.2 Absolute values give distances between points on the number line.

Since the symbol $\sqrt{a}$ always denotes the *nonnegative* square root of a, an alternate definition of $|x|$ is

$$|x| = \sqrt{x^2}.$$

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

The absolute value has the following properties. (You are asked to prove these properties in the exercises.)

Absolute Value Properties

1. $|-a| = |a|$ — A number and its additive inverse or negative have the same absolute value.
2. $|ab| = |a||b|$ — The absolute value of a product is the product of the absolute values.
3. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$ — The absolute value of a quotient is the quotient of the absolute values.
4. $|a + b| \leq |a| + |b|$ — The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Note that $|-a| \neq -|a|$. For example, $|-3| = 3$, whereas $-|3| = -3$. If a and b differ in sign, then $|a + b|$ is less than $|a| + |b|$. In all other cases, $|a + b|$ equals $|a| + |b|$. Absolute value bars in expressions like $|-3 + 5|$ work like parentheses: We do the arithmetic inside *before* taking the absolute value.

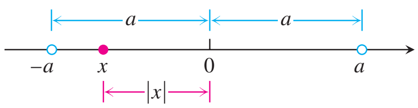


FIGURE 1.3 $|x| < a$ means x lies between $-a$ and a.

EXAMPLE 3 Illustrating the Triangle Inequality

$$|-3 + 5| = |2| = 2 < |-3| + |5| = 8$$
$$|3 + 5| = |8| = |3| + |5|$$
$$|-3 - 5| = |-8| = 8 = |-3| + |-5|$$
■

The inequality $|x| < a$ says that the distance from x to 0 is less than the positive number a. This means that x must lie between $-a$ and a, as we can see from Figure 1.3.

The following statements are all consequences of the definition of absolute value and are often helpful when solving equations or inequalities involving absolute values.

> **Absolute Values and Intervals**
> If a is any positive number, then
>
> **5.** $|x| = a$ if and only if $x = \pm a$
> **6.** $|x| < a$ if and only if $-a < x < a$
> **7.** $|x| > a$ if and only if $x > a$ or $x < -a$
> **8.** $|x| \leq a$ if and only if $-a \leq x \leq a$
> **9.** $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

The symbol $\Leftrightarrow$ is often used by mathematicians to denote the "if and only if" logical relationship. It also means "implies and is implied by."

EXAMPLE 4 Solving an Equation with Absolute Values

Solve the equation $|2x - 3| = 7$.

Solution By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$$\begin{aligned} 2x - 3 &= 7 & 2x - 3 &= -7 \\ 2x &= 10 & 2x &= -4 \\ x &= 5 & x &= -2 \end{aligned}$$

Equivalent equations without absolute values

Solve as usual.

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$. ■

EXAMPLE 5 Solving an Inequality Involving Absolute Values

Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$.

Solution We have

$$\left|5-\frac{2}{x}\right|<1 \Leftrightarrow -1<5-\frac{2}{x}<1 \qquad \text{Property 6}$$
$$\Leftrightarrow -6<-\frac{2}{x}<-4 \qquad \text{Subtract 5.}$$
$$\Leftrightarrow 3>\frac{1}{x}>2 \qquad \text{Multiply by } -\tfrac{1}{2}.$$
$$\Leftrightarrow \frac{1}{3}<x<\frac{1}{2}. \qquad \text{Take reciprocals.}$$

Notice how the various rules for inequalities were used here. Multiplying by a negative number reverses the inequality. So does taking reciprocals in an inequality in which both sides are positive. The original inequality holds if and only if $(1/3)<x<(1/2)$. The solution set is the open interval $(1/3, 1/2)$. ■

EXAMPLE 6 Solve the inequality and show the solution set on the real line:

(a) $|2x-3|\le 1$ **(b)** $|2x-3|\ge 1$

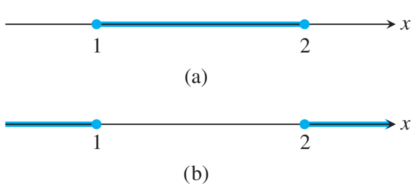


FIGURE 1.4 The solution sets (a) $[1, 2]$ and (b) $(-\infty, 1]\cup[2, \infty)$ in Example 6.

Solution

(a)
$$|2x-3|\le 1$$
$$-1\le 2x-3\le 1 \qquad \text{Property 8}$$
$$2\le 2x\le 4 \qquad \text{Add 3.}$$
$$1\le x\le 2 \qquad \text{Divide by 2.}$$

The solution set is the closed interval $[1, 2]$ (Figure 1.4a).

(b)
$$|2x-3|\ge 1$$
$$2x-3\ge 1 \quad \text{or} \quad 2x-3\le -1 \qquad \text{Property 9}$$
$$x-\frac{3}{2}\ge\frac{1}{2} \quad \text{or} \quad x-\frac{3}{2}\le -\frac{1}{2} \qquad \text{Divide by 2.}$$
$$x\ge 2 \quad \text{or} \quad x\le 1 \qquad \text{Add } \tfrac{3}{2}.$$

The solution set is $(-\infty, 1]\cup[2, \infty)$ (Figure 1.4b). ■

EXERCISES 1.1

Decimal Representations

1. Express $1/9$ as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of $2/9$? $3/9$? $8/9$? $9/9$?
2. Express $1/11$ as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of $2/11$? $3/11$? $9/11$? $11/11$?

Inequalities

3. If $2<x<6$, which of the following statements about x are necessarily true, and which are not necessarily true?
   a. $0<x<4$ **b.** $0<x-2<4$
   c. $1<\frac{x}{2}<3$ **d.** $\frac{1}{6}<\frac{1}{x}<\frac{1}{2}$
   e. $1<\frac{6}{x}<3$ **f.** $|x-4|<2$
   g. $-6<-x<2$ **h.** $-6<-x<-2$

4. If $-1 < y - 5 < 1$, which of the following statements about y are necessarily true, and which are not necessarily true?

a. $4 < y < 6$ **b.** $-6 < y < -4$

c. $y > 4$ **d.** $y < 6$

e. $0 < y - 4 < 2$ **f.** $2 < \frac{y}{2} < 3$

g. $\frac{1}{6} < \frac{1}{y} < \frac{1}{4}$ **h.** $|y - 5| < 1$

In Exercises 5–12, solve the inequalities and show the solution sets on the real line.

5. $-2x > 4$ **6.** $8 - 3x \geq 5$

7. $5x - 3 \leq 7 - 3x$ **8.** $3(2 - x) > 2(3 + x)$

9. $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$ **10.** $\frac{6 - x}{4} < \frac{3x - 4}{2}$

11. $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$ **12.** $-\frac{x + 5}{2} \leq \frac{12 + 3x}{4}$

Absolute Value

Solve the equations in Exercises 13–18.

13. $|y| = 3$ **14.** $|y - 3| = 7$ **15.** $|2t + 5| = 4$

16. $|1 - t| = 1$ **17.** $|8 - 3s| = \frac{9}{2}$ **18.** $\left|\frac{s}{2} - 1\right| = 1$

Solve the inequalities in Exercises 19–34, expressing the solution sets as intervals or unions of intervals. Also, show each solution set on the real line.

19. $|x| < 2$ **20.** $|x| \leq 2$ **21.** $|t - 1| \leq 3$

22. $|t + 2| < 1$ **23.** $|3y - 7| < 4$ **24.** $|2y + 5| < 1$

25. $\left|\frac{z}{5} - 1\right| \leq 1$ **26.** $\left|\frac{3}{2}z - 1\right| \leq 2$ **27.** $\left|3 - \frac{1}{x}\right| < \frac{1}{2}$

28. $\left|\frac{2}{x} - 4\right| < 3$ **29.** $|2s| \geq 4$ **30.** $|s + 3| \geq \frac{1}{2}$

31. $|1 - x| > 1$ **32.** $|2 - 3x| > 5$ **33.** $\left|\frac{r + 1}{2}\right| \geq 1$

34. $\left|\frac{3r}{5} - 1\right| > \frac{2}{5}$

Quadratic Inequalities

Solve the inequalities in Exercises 35–42. Express the solution sets as intervals or unions of intervals and show them on the real line. Use the result $\sqrt{a^2} = |a|$ as appropriate.

35. $x^2 < 2$ **36.** $4 \leq x^2$ **37.** $4 < x^2 < 9$

38. $\frac{1}{9} < x^2 < \frac{1}{4}$ **39.** $(x - 1)^2 < 4$ **40.** $(x + 3)^2 < 2$

41. $x^2 - x < 0$ **42.** $x^2 - x - 2 \geq 0$

Theory and Examples

43. Do not fall into the trap $|-a| = a$. For what real numbers a is this equation true? For what real numbers is it false?

44. Solve the equation $|x - 1| = 1 - x$.

45. **A proof of the triangle inequality** Give the reason justifying each of the numbered steps in the following proof of the triangle inequality.

$$\begin{aligned}
|a + b|^2 &= (a + b)^2 && (1)\\
&= a^2 + 2ab + b^2 \\
&\leq a^2 + 2|a||b| + b^2 && (2)\\
&= |a|^2 + 2|a||b| + |b|^2 && (3)\\
&= (|a| + |b|)^2 \\
|a + b| &\leq |a| + |b| && (4)
\end{aligned}$$

46. Prove that $|ab| = |a||b|$ for any numbers a and b.

47. If $|x| \leq 3$ and $x > -1/2$, what can you say about x?

48. Graph the inequality $|x| + |y| \leq 1$.

49. Let $f(x) = 2x + 1$ and let $\delta > 0$ be any positive number. Prove that $|x - 1| < \delta$ implies $|f(x) - f(1)| < 2\delta$. Here the notation $f(a)$ means the value of the expression $2x + 1$ when $x = a$. This *function notation* is explained in Section 1.3.

50. Let $f(x) = 2x + 3$ and let $\epsilon > 0$ be any positive number. Prove that $|f(x) - f(0)| < \epsilon$ whenever $|x - 0| < \frac{\epsilon}{2}$. Here the notation $f(a)$ means the value of the expression $2x + 3$ when $x = a$. (See Section 1.3.)

51. For any number a, prove that $|-a| = |a|$.

52. Let a be any positive number. Prove that $|x| > a$ if and only if $x > a$ or $x < -a$.

53. **a.** If b is any nonzero real number, prove that $|1/b| = 1/|b|$.

b. Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for any numbers a and $b \neq 0$.

54. Using mathematical induction (see Appendix 1), prove that $|a^n| = |a|^n$ for any number a and positive integer n.

1.2 Lines, Circles, and Parabolas

This section reviews coordinates, lines, distance, circles, and parabolas in the plane. The notion of increment is also discussed.

Cartesian Coordinates in the Plane

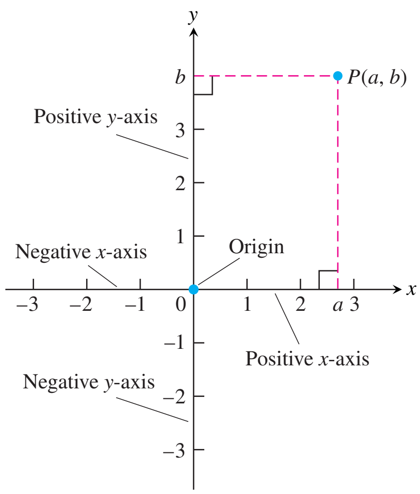


FIGURE 1.5 Cartesian coordinates in the plane are based on two perpendicular axes intersecting at the origin.

In the previous section we identified the points on the line with real numbers by assigning them coordinates. Points in the plane can be identified with ordered pairs of real numbers. To begin, we draw two perpendicular coordinate lines that intersect at the 0-point of each line. These lines are called **coordinate axes** in the plane. On the horizontal x-axis, numbers are denoted by x and increase to the right. On the vertical y-axis, numbers are denoted by y and increase upward (Figure 1.5). Thus "upward" and "to the right" are positive directions, whereas "downward" and "to the left" are considered as negative. The **origin** O, also labeled 0, of the coordinate system is the point in the plane where x and y are both zero.

If P is any point in the plane, it can be located by exactly one ordered pair of real numbers in the following way. Draw lines through P perpendicular to the two coordinate axes. These lines intersect the axes at points with coordinates a and b (Figure 1.5). The ordered pair (a, b) is assigned to the point P and is called its **coordinate pair**. The first number a is the ***x*-coordinate** (or **abscissa**) of P; the second number b is the ***y*-coordinate** (or **ordinate**) of P. The x-coordinate of every point on the y-axis is 0. The y-coordinate of every point on the x-axis is 0. The origin is the point $(0, 0)$.

Starting with an ordered pair (a, b), we can reverse the process and arrive at a corresponding point P in the plane. Often we identify P with the ordered pair and write $P(a, b)$. We sometimes also refer to "the point (a, b)" and it will be clear from the context when (a, b) refers to a point in the plane and not to an open interval on the real line. Several points labeled by their coordinates are shown in Figure 1.6.

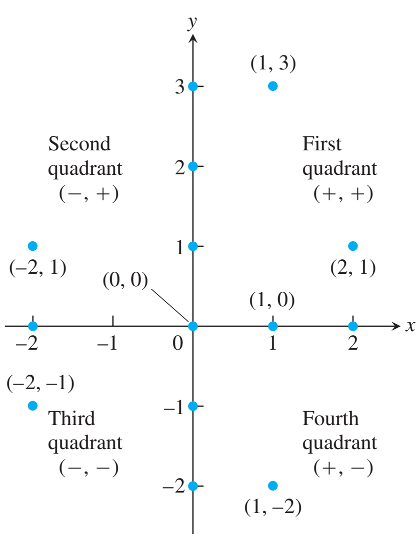


FIGURE 1.6 Points labeled in the xy-coordinate or Cartesian plane. The points on the axes all have coordinate pairs but are usually labeled with single real numbers, (so $(1, 0)$ on the x-axis is labeled as 1). Notice the coordinate sign patterns of the quadrants.

This coordinate system is called the **rectangular coordinate system** or **Cartesian coordinate system** (after the sixteenth century French mathematician René Descartes). The coordinate axes of this coordinate or Cartesian plane divide the plane into four regions called **quadrants**, numbered counterclockwise as shown in Figure 1.6.

The **graph** of an equation or inequality in the variables x and y is the set of all points $P(x, y)$ in the plane whose coordinates satisfy the equation or inequality. When we plot data in the coordinate plane or graph formulas whose variables have different units of measure, we do not need to use the same scale on the two axes. If we plot time vs. thrust for a rocket motor, for example, there is no reason to place the mark that shows 1 sec on the time axis the same distance from the origin as the mark that shows 1 lb on the thrust axis.

Usually when we graph functions whose variables do not represent physical measurements and when we draw figures in the coordinate plane to study their geometry and trigonometry, we try to make the scales on the axes identical. A vertical unit of distance then looks the same as a horizontal unit. As on a surveyor's map or a scale drawing, line segments that are supposed to have the same length will look as if they do and angles that are supposed to be congruent will look congruent.

Computer displays and calculator displays are another matter. The vertical and horizontal scales on machine-generated graphs usually differ, and there are corresponding distortions in distances, slopes, and angles. Circles may look like ellipses, rectangles may look like squares, right angles may appear to be acute or obtuse, and so on. We discuss these displays and distortions in greater detail in Section 1.7.

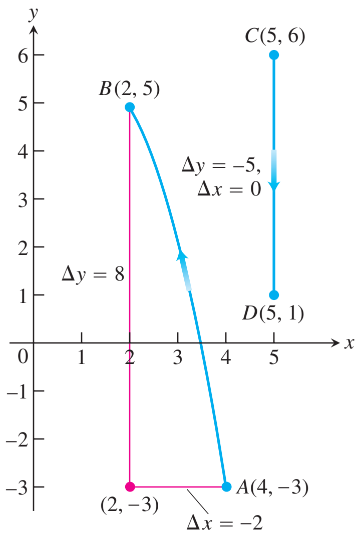


FIGURE 1.7 Coordinate increments may be positive, negative, or zero (Example 1).

HISTORICAL BIOGRAPHY*

René Descartes
(1596–1650)

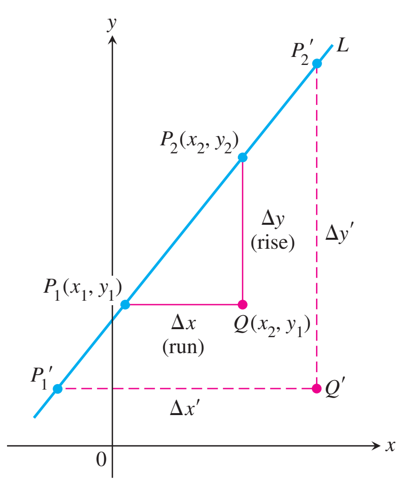


FIGURE 1.8 Triangles P_1QP_2 and $P_1'Q'P_2'$ are similar, so the ratio of their sides has the same value for any two points on the line. This common value is the line's slope.

Increments and Straight Lines

When a particle moves from one point in the plane to another, the net changes in its coordinates are called *increments.* They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. If x changes from x_1 to x_2, the **increment** in x is

$$\Delta x = x_2 - x_1.$$

EXAMPLE 1 In going from the point $A(4, -3)$ to the point $B(2, 5)$ the increments in the x- and y-coordinates are

$$\Delta x = 2 - 4 = -2, \qquad \Delta y = 5 - (-3) = 8.$$

From $C(5, 6)$ to $D(5, 1)$ the coordinate increments are

$$\Delta x = 5 - 5 = 0, \qquad \Delta y = 1 - 6 = -5.$$

See Figure 1.7. ■

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ the **run** and the **rise**, respectively, between P_1 and P_2. Two such points always determine a unique straight line (usually called simply a line) passing through them both. We call the line P_1P_2.

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

has the same value for every choice of the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line (Figure 1.8). This is because the ratios of corresponding sides for similar triangles are equal.

DEFINITION Slope

The constant

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the **slope** of the nonvertical line P_1P_2.

The slope tells us the direction (uphill, downhill) and steepness of a line. A line with positive slope rises uphill to the right; one with negative slope falls downhill to the right (Figure 1.9). The greater the absolute value of the slope, the more rapid the rise or fall. The slope of a vertical line is *undefined*. Since the run Δx is zero for a vertical line, we cannot evaluate the slope ratio m.

The direction and steepness of a line can also be measured with an angle. The **angle of inclination** of a line that crosses the x-axis is the smallest counterclockwise angle from the x-axis to the line (Figure 1.10). The inclination of a horizontal line is 0°. The inclination of a vertical line is 90°. If ϕ (the Greek letter phi) is the inclination of a line, then $0 \le \phi < 180°$.

*To learn more about the historical figures and the development of the major elements and topics of calculus, visit **www.aw-bc.com/thomas**.

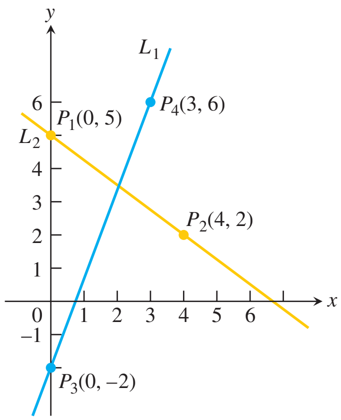


FIGURE 1.9 The slope of L_1 is

$$m = \frac{\Delta y}{\Delta x} = \frac{6 - (-2)}{3 - 0} = \frac{8}{3}.$$

That is, y increases 8 units every time x increases 3 units. The slope of L_2 is

$$m = \frac{\Delta y}{\Delta x} = \frac{2 - 5}{4 - 0} = \frac{-3}{4}.$$

That is, y decreases 3 units every time x increases 4 units.

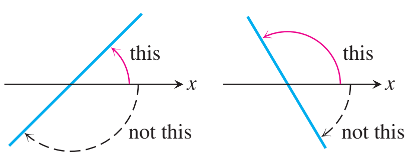


FIGURE 1.10 Angles of inclination are measured counterclockwise from the x-axis.

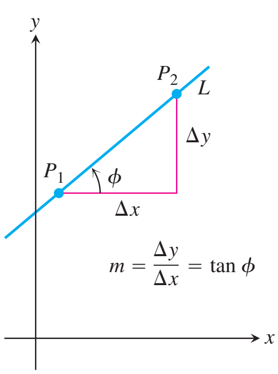


FIGURE 1.11 The slope of a nonvertical line is the tangent of its angle of inclination.

The relationship between the slope m of a nonvertical line and the line's angle of inclination ϕ is shown in Figure 1.11:

$$m = \tan\phi.$$

Straight lines have relatively simple equations. All points on the *vertical line* through the point a on the x-axis have x-coordinates equal to a. Thus, $x = a$ is an equation for the vertical line. Similarly, $y = b$ is an equation for the *horizontal line* meeting the y-axis at b. (See Figure 1.12.)

We can write an equation for a nonvertical straight line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is *any* other point on L, then we can use the two points P_1 and P to compute the slope,

$$m = \frac{y - y_1}{x - x_1}$$

so that

$$y - y_1 = m(x - x_1) \qquad \text{or} \qquad y = y_1 + m(x - x_1).$$

> The equation
>
> $$y = y_1 + m(x - x_1)$$
>
> is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m.

EXAMPLE 2 Write an equation for the line through the point (2, 3) with slope $-3/2$.

Solution We substitute $x_1 = 2$, $y_1 = 3$, and $m = -3/2$ into the point-slope equation and obtain

$$y = 3 - \frac{3}{2}\left(x - 2\right), \qquad \text{or} \qquad y = -\frac{3}{2}x + 6.$$

When $x = 0$, $y = 6$ so the line intersects the y-axis at $y = 6$. ■

EXAMPLE 3 A Line Through Two Points

Write an equation for the line through $(-2, -1)$ and $(3, 4)$.

Solution The line's slope is

$$m = \frac{-1 - 4}{-2 - 3} = \frac{-5}{-5} = 1.$$

We can use this slope with either of the two given points in the point-slope equation:

With $(x_1, y_1) = (-2, -1)$	**With $(x_1, y_1) = (3, 4)$**
$y = -1 + 1 \cdot (x - (-2))$	$y = 4 + 1 \cdot (x - 3)$
$y = -1 + x + 2$	$y = 4 + x - 3$
$y = x + 1$	$y = x + 1$

Same result

Either way, $y = x + 1$ is an equation for the line (Figure 1.13). ■

The y-coordinate of the point where a nonvertical line intersects the y-axis is called the ***y*-intercept** of the line. Similarly, the ***x*-intercept** of a nonhorizontal line is the x-coordinate of the point where it crosses the x-axis (Figure 1.14). A line with slope m and y-intercept b passes through the point $(0, b)$, so it has equation

$$y = b + m(x - 0), \qquad \text{or, more simply,} \qquad y = mx + b.$$

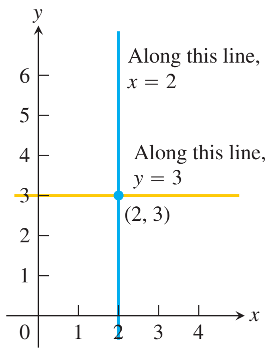


FIGURE 1.12 The standard equations for the vertical and horizontal lines through (2, 3) are $x = 2$ and $y = 3$.

> The equation
>
> $$y = mx + b$$
>
> is called the **slope-intercept equation** of the line with slope m and y-intercept b.

Lines with equations of the form $y = mx$ have y-intercept 0 and so pass through the origin. Equations of lines are called **linear** equations.

The equation

$$Ax + By = C \qquad (A \text{ and } B \text{ not both } 0)$$

is called the **general linear equation** in x and y because its graph always represents a line and every line has an equation in this form (including lines with undefined slope).

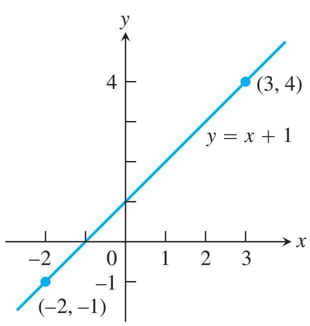


FIGURE 1.13 The line in Example 3.

EXAMPLE 4 Finding the Slope and y-Intercept

Find the slope and y-intercept of the line $8x + 5y = 20$.

Solution Solve the equation for y to put it in slope-intercept form:

$$\begin{aligned} 8x + 5y &= 20 \\ 5y &= -8x + 20 \\ y &= -\frac{8}{5}x + 4. \end{aligned}$$

The slope is $m = -8/5$. The y-intercept is $b = 4$. ■

Parallel and Perpendicular Lines

Lines that are parallel have equal angles of inclination, so they have the same slope (if they are not vertical). Conversely, lines with equal slopes have equal angles of inclination and so are parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}.$$

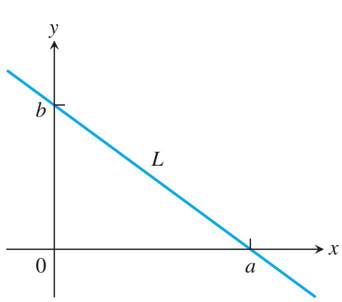


FIGURE 1.14 Line L has x-intercept a and y-intercept b.

To see this, notice by inspecting similar triangles in Figure 1.15 that $m_1 = a/h$, and $m_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

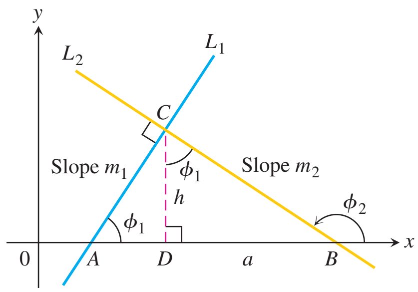


FIGURE 1.15 ΔADC is similar to ΔCDB. Hence ϕ_1 is also the upper angle in ΔCDB. From the sides of ΔCDB, we read $\tan \phi_1 = a/h$.

Distance and Circles in the Plane

The distance between points in the plane is calculated with a formula that comes from the Pythagorean theorem (Figure 1.16).

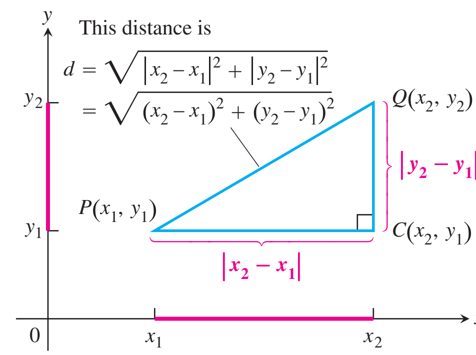


FIGURE 1.16 To calculate the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$, apply the Pythagorean theorem to triangle PCQ.

Distance Formula for Points in the Plane

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 5 Calculating Distance

(a) The distance between $P(-1, 2)$ and $Q(3, 4)$ is

$$\sqrt{(3 - (-1))^2 + (4 - 2)^2} = \sqrt{(4)^2 + (2)^2} = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}.$$

(b) The distance from the origin to $P(x, y)$ is

$$\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}.$$

By definition, a **circle** of radius a is the set of all points $P(x, y)$ whose distance from some center $C(h, k)$ equals a (Figure 1.17). From the distance formula, P lies on the circle if and only if

$$\sqrt{(x - h)^2 + (y - k)^2} = a,$$

so

$$(x - h)^2 + (y - k)^2 = a^2. \tag{1}$$

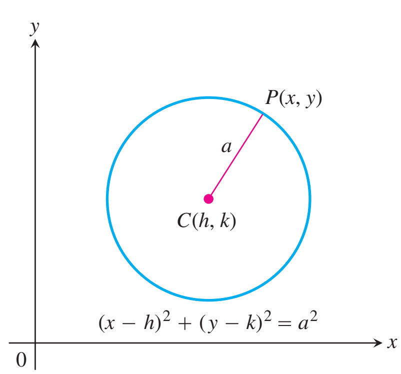


FIGURE 1.17 A circle of radius a in the xy-plane, with center at (h, k).

Equation (1) is the **standard equation** of a circle with center (h, k) and radius a. The circle of radius $a = 1$ and centered at the origin is the **unit circle** with equation

$$x^2 + y^2 = 1.$$

EXAMPLE 6

(a) The standard equation for the circle of radius 2 centered at (3, 4) is

$$(x-3)^2+(y-4)^2=2^2=4.$$

(b) The circle

$$(x-1)^2+(y+5)^2=3$$

has $h=1, k=-5$, and $a=\sqrt{3}$. The center is the point $(h,k)=(1,-5)$ and the radius is $a=\sqrt{3}$. ■

If an equation for a circle is not in standard form, we can find the circle's center and radius by first converting the equation to standard form. The algebraic technique for doing so is *completing the square* (see Appendix 9).

EXAMPLE 7 Finding a Circle's Center and Radius

Find the center and radius of the circle

$$x^2+y^2+4x-6y-3=0.$$

Solution We convert the equation to standard form by completing the squares in x and y:

$$x^2+y^2+4x-6y-3=0$$

Start with the given equation.

$$(x^2+4x\quad)+(y^2-6y\quad)=3$$

Gather terms. Move the constant to the right-hand side.

$$\left(x^2+4x+\left(\frac{4}{2}\right)^2\right)+\left(y^2-6y+\left(\frac{-6}{2}\right)^2\right)=3+\left(\frac{4}{2}\right)^2+\left(\frac{-6}{2}\right)^2$$

Add the square of half the coefficient of x to each side of the equation. Do the same for y. The parenthetical expressions on the left-hand side are now perfect squares.

$$(x^2+4x+4)+(y^2-6y+9)=3+4+9$$

$$(x+2)^2+(y-3)^2=16$$

Write each quadratic as a squared linear expression.

The center is $(-2,3)$ and the radius is $a=4$. ■

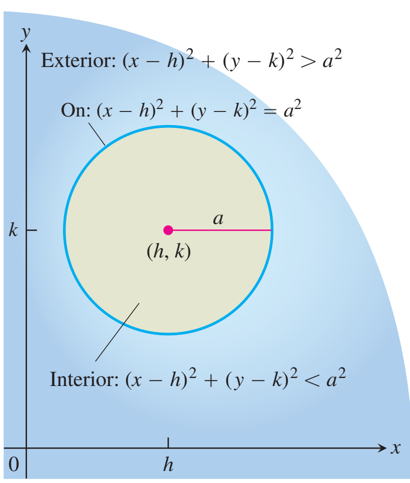


FIGURE 1.18 The interior and exterior of the circle $(x-h)^2+(y-k)^2=a^2$.

The points (x, y) satisfying the inequality

$$(x-h)^2+(y-k)^2<a^2$$

make up the **interior** region of the circle with center (h, k) and radius a (Figure 1.18). The circle's **exterior** consists of the points (x, y) satisfying

$$(x-h)^2+(y-k)^2>a^2.$$

Parabolas

The geometric definition and properties of general parabolas are reviewed in Section 10.1. Here we look at parabolas arising as the graphs of equations of the form $y=ax^2+bx+c$.

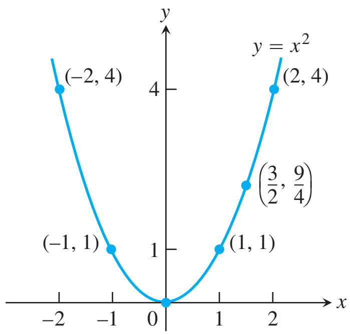


FIGURE 1.19 The parabola $y = x^2$ (Example 8).

EXAMPLE 8 The Parabola $y = x^2$

Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0)$, $(1, 1)$, $\left(\frac{3}{2}, \frac{9}{4}\right)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$. These points (and all others satisfying the equation) make up a smooth curve called a parabola (Figure 1.19). ■

The graph of an equation of the form

$$y = ax^2$$

is a **parabola** whose **axis** (axis of symmetry) is the y-axis. The parabola's **vertex** (point where the parabola and axis cross) lies at the origin. The parabola opens upward if $a > 0$ and downward if $a < 0$. The larger the value of $|a|$, the narrower the parabola (Figure 1.20).

Generally, the graph of $y = ax^2 + bx + c$ is a shifted and scaled version of the parabola $y = x^2$. We discuss shifting and scaling of graphs in more detail in Section 1.5.

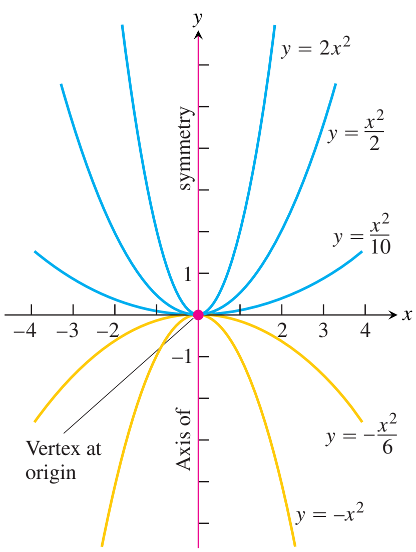


FIGURE 1.20 Besides determining the direction in which the parabola $y = ax^2$ opens, the number a is a scaling factor. The parabola widens as a approaches zero and narrows as $|a|$ becomes large.

The Graph of $y = ax^2 + bx + c$, $a \neq 0$

The graph of the equation $y = ax^2 + bx + c$, $a \neq 0$, is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. The **axis** is the line

$$x = -\frac{b}{2a}. \tag{2}$$

The **vertex** of the parabola is the point where the axis and parabola intersect. Its x-coordinate is $x = -b/2a$; its y-coordinate is found by substituting $x = -b/2a$ in the parabola's equation.

Notice that if $a = 0$, then we have $y = bx + c$ which is an equation for a line. The axis, given by Equation (2), can be found by completing the square or by using a technique we study in Section 4.1.

EXAMPLE 9 Graphing a Parabola

Graph the equation $y = -\frac{1}{2}x^2 - x + 4$.

Solution Comparing the equation with $y = ax^2 + bx + c$ we see that

$$a = -\frac{1}{2}, \qquad b = -1, \qquad c = 4.$$

Since $a < 0$, the parabola opens downward. From Equation (2) the axis is the vertical line

$$x = -\frac{b}{2a} = -\frac{(-1)}{2(-1/2)} = -1.$$

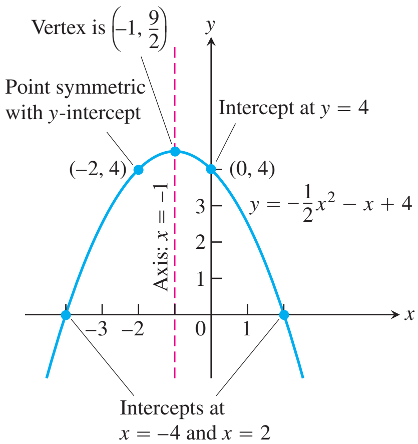


FIGURE 1.21 The parabola in Example 9.

When $x = -1$, we have

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}.$$

The vertex is $(-1, 9/2)$.

The x-intercepts are where $y = 0$:

$$\begin{aligned} -\frac{1}{2}x^2 - x + 4 &= 0 \\ x^2 + 2x - 8 &= 0 \\ (x - 2)(x + 4) &= 0 \\ x = 2, \qquad x &= -4 \end{aligned}$$

We plot some points, sketch the axis, and use the direction of opening to complete the graph in Figure 1.21. ■

EXERCISES 1.2

Increments and Distance

In Exercises 1–4, a particle moves from A to B in the coordinate plane. Find the increments Δx and Δy in the particle's coordinates. Also find the distance from A to B.

1. $A(-3, 2)$, $B(-1, -2)$ **2.** $A(-1, -2)$, $B(-3, 2)$

3. $A(-3.2, -2)$, $B(-8.1, -2)$ **4.** $A(\sqrt{2}, 4)$, $B(0, 1.5)$

Describe the graphs of the equations in Exercises 5–8.

5. $x^2 + y^2 = 1$ **6.** $x^2 + y^2 = 2$

7. $x^2 + y^2 \le 3$ **8.** $x^2 + y^2 = 0$

Slopes, Lines, and Intercepts

Plot the points in Exercises 9–12 and find the slope (if any) of the line they determine. Also find the common slope (if any) of the lines perpendicular to line AB.

9. $A(-1, 2)$, $B(-2, -1)$ **10.** $A(-2, 1)$, $B(2, -2)$

11. $A(2, 3)$, $B(-1, 3)$ **12.** $A(-2, 0)$, $B(-2, -2)$

In Exercises 13–16, find an equation for (a) the vertical line and (b) the horizontal line through the given point.

13. $(-1, 4/3)$ **14.** $\left(\sqrt{2}, -1.3\right)$

15. $\left(0, -\sqrt{2}\right)$ **16.** $(-\pi, 0)$

In Exercises 17–30, write an equation for each line described.

17. Passes through $(-1, 1)$ with slope -1

18. Passes through $(2, -3)$ with slope $1/2$

19. Passes through $(3, 4)$ and $(-2, 5)$

20. Passes through $(-8, 0)$ and $(-1, 3)$

21. Has slope $-5/4$ and y-intercept 6

22. Has slope $1/2$ and y-intercept -3

23. Passes through $(-12, -9)$ and has slope 0

24. Passes through $(1/3, 4)$, and has no slope

25. Has y-intercept 4 and x-intercept -1

26. Has y-intercept -6 and x-intercept 2

27. Passes through $(5, -1)$ and is parallel to the line $2x + 5y = 15$

28. Passes through $\left(-\sqrt{2}, 2\right)$ parallel to the line $\sqrt{2}x + 5y = \sqrt{3}$

29. Passes through $(4, 10)$ and is perpendicular to the line $6x - 3y = 5$

30. Passes through $(0, 1)$ and is perpendicular to the line $8x - 13y = 13$

In Exercises 31–34, find the line's x- and y-intercepts and use this information to graph the line.

31. $3x + 4y = 12$ **32.** $x + 2y = -4$

33. $\sqrt{2}x - \sqrt{3}y = \sqrt{6}$ **34.** $1.5x - y = -3$

35. Is there anything special about the relationship between the lines $Ax + By = C_1$ and $Bx - Ay = C_2$ $(A \neq 0, B \neq 0)$? Give reasons for your answer.

36. Is there anything special about the relationship between the lines $Ax + By = C_1$ and $Ax + By = C_2$ $(A \neq 0, B \neq 0)$? Give reasons for your answer.

Increments and Motion

37. A particle starts at $A(-2, 3)$ and its coordinates change by increments $\Delta x = 5$, $\Delta y = -6$. Find its new position.

38. A particle starts at $A(6, 0)$ and its coordinates change by increments $\Delta x = -6$, $\Delta y = 0$. Find its new position.

39. The coordinates of a particle change by $\Delta x = 5$ and $\Delta y = 6$ as it moves from $A(x, y)$ to $B(3, -3)$. Find x and y.

40. A particle started at $A(1, 0)$, circled the origin once counterclockwise, and returned to $A(1, 0)$. What were the net changes in its coordinates?

Circles

In Exercises 41–46, find an equation for the circle with the given center $C(h, k)$ and radius a. Then sketch the circle in the xy-plane. Include the circle's center in your sketch. Also, label the circle's x- and y-intercepts, if any, with their coordinate pairs.

41. $C(0, 2), \quad a = 2$

42. $C(-3, 0), \quad a = 3$

43. $C(-1, 5), \quad a = \sqrt{10}$

44. $C(1, 1), \quad a = \sqrt{2}$

45. $C(-\sqrt{3}, -2), \quad a = 2$

46. $C(3, 1/2), \quad a = 5$

Graph the circles whose equations are given in Exercises 47–52. Label each circle's center and intercepts (if any) with their coordinate pairs.

47. $x^2 + y^2 + 4x - 4y + 4 = 0$

48. $x^2 + y^2 - 8x + 4y + 16 = 0$

49. $x^2 + y^2 - 3y - 4 = 0$

50. $x^2 + y^2 - 4x - (9/4) = 0$

51. $x^2 + y^2 - 4x + 4y = 0$

52. $x^2 + y^2 + 2x = 3$

Parabolas

Graph the parabolas in Exercises 53–60. Label the vertex, axis, and intercepts in each case.

53. $y = x^2 - 2x - 3$

54. $y = x^2 + 4x + 3$

55. $y = -x^2 + 4x$

56. $y = -x^2 + 4x - 5$

57. $y = -x^2 - 6x - 5$

58. $y = 2x^2 - x + 3$

59. $y = \frac{1}{2}x^2 + x + 4$

60. $y = -\frac{1}{4}x^2 + 2x + 4$

Inequalities

Describe the regions defined by the inequalities and pairs of inequalities in Exercises 61–68.

61. $x^2 + y^2 > 7$

62. $x^2 + y^2 < 5$

63. $(x - 1)^2 + y^2 \le 4$

64. $x^2 + (y - 2)^2 \ge 4$

65. $x^2 + y^2 > 1, \quad x^2 + y^2 < 4$

66. $x^2 + y^2 \le 4, \quad (x + 2)^2 + y^2 \le 4$

67. $x^2 + y^2 + 6y < 0, \quad y > -3$

68. $x^2 + y^2 - 4x + 2y > 4, \quad x > 2$

69. Write an inequality that describes the points that lie inside the circle with center $(-2, 1)$ and radius $\sqrt{6}$.

70. Write an inequality that describes the points that lie outside the circle with center $(-4, 2)$ and radius 4.

71. Write a pair of inequalities that describe the points that lie inside or on the circle with center (0, 0) and radius $\sqrt{2}$, and on or to the right of the vertical line through (1, 0).

72. Write a pair of inequalities that describe the points that lie outside the circle with center (0, 0) and radius 2, and inside the circle that has center (1, 3) and passes through the origin.

Intersecting Lines, Circles, and Parabolas

In Exercises 73–80, graph the two equations and find the points in which the graphs intersect.

73. $y = 2x, \quad x^2 + y^2 = 1$

74. $x + y = 1, \quad (x - 1)^2 + y^2 = 1$

75. $y - x = 1, \quad y = x^2$

76. $x + y = 0, \quad y = -(x - 1)^2$

77. $y = -x^2, \quad y = 2x^2 - 1$

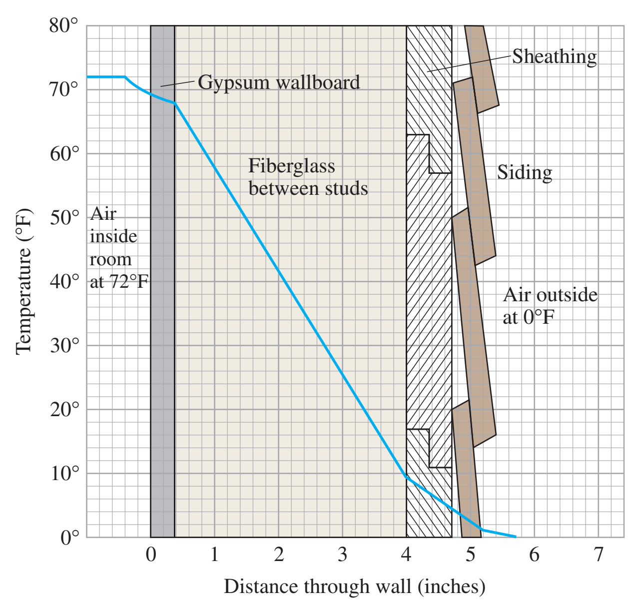
78. $y = \frac{1}{4}x^2, \quad y = (x - 1)^2$

79. $x^2 + y^2 = 1, \quad (x - 1)^2 + y^2 = 1$

80. $x^2 + y^2 = 1, \quad x^2 + y = 1$

Applications

81. Insulation By measuring slopes in the accompanying figure, estimate the temperature change in degrees per inch for (a) the gypsum wallboard; (b) the fiberglass insulation; (c) the wood sheathing.

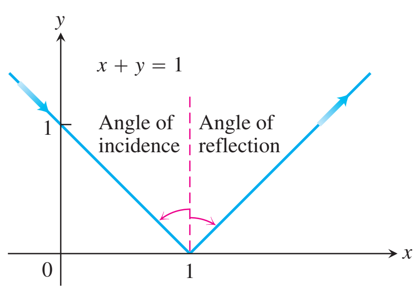


The temperature changes in the wall in Exercises 81 and 82.

82. Insulation According to the figure in Exercise 81, which of the materials is the best insulator? the poorest? Explain.

83. Pressure under water The pressure p experienced by a diver under water is related to the diver's depth d by an equation of the form $p = kd + 1$ (k a constant). At the surface, the pressure is 1 atmosphere. The pressure at 100 meters is about 10.94 atmospheres. Find the pressure at 50 meters.

84. Reflected light A ray of light comes in along the line $x + y = 1$ from the second quadrant and reflects off the x-axis (see the accompanying figure). The angle of incidence is equal to the angle of reflection. Write an equation for the line along which the departing light travels.



The path of the light ray in Exercise 84. Angles of incidence and reflection are measured from the perpendicular.

85. Fahrenheit vs. Celsius In the FC-plane, sketch the graph of the equation

$$C = \frac{5}{9}(F - 32)$$

linking Fahrenheit and Celsius temperatures. On the same graph sketch the line $C = F$. Is there a temperature at which a Celsius thermometer gives the same numerical reading as a Fahrenheit thermometer? If so, find it.

86. The Mt. Washington Cog Railway Civil engineers calculate the slope of roadbed as the ratio of the distance it rises or falls to the distance it runs horizontally. They call this ratio the **grade** of the roadbed, usually written as a percentage. Along the coast, commercial railroad grades are usually less than 2%. In the mountains, they may go as high as 4%. Highway grades are usually less than 5%.

The steepest part of the Mt. Washington Cog Railway in New Hampshire has an exceptional 37.1% grade. Along this part of the track, the seats in the front of the car are 14 ft above those in the rear. About how far apart are the front and rear rows of seats?

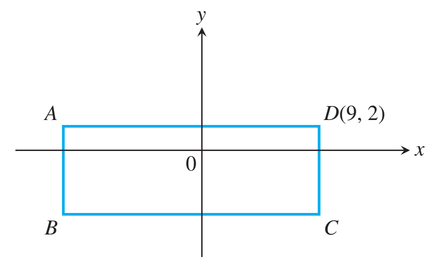
Theory and Examples

87. By calculating the lengths of its sides, show that the triangle with vertices at the points $A(1, 2)$, $B(5, 5)$, and $C(4, -2)$ is isosceles but not equilateral.

88. Show that the triangle with vertices $A(0, 0)$, $B\left(1, \sqrt{3}\right)$, and $C(2, 0)$ is equilateral.

89. Show that the points $A(2, -1)$, $B(1, 3)$, and $C(-3, 2)$ are vertices of a square, and find the fourth vertex.

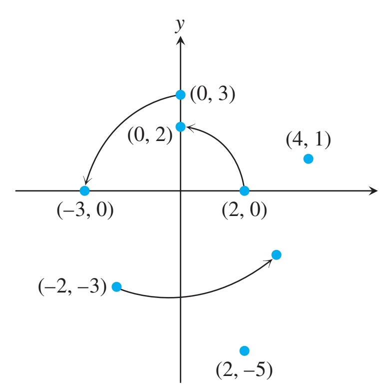
90. The rectangle shown here has sides parallel to the axes. It is three times as long as it is wide, and its perimeter is 56 units. Find the coordinates of the vertices A, B, and C.



91. Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Sketch them and find the coordinates of the fourth vertex of each.

92. A 90° rotation counterclockwise about the origin takes $(2, 0)$ to $(0, 2)$, and $(0, 3)$ to $(-3, 0)$, as shown in the accompanying figure. Where does it take each of the following points?

a. $(4, 1)$ **b.** $(-2, -3)$ **c.** $(2, -5)$
d. $(x, 0)$ **e.** $(0, y)$ **f.** (x, y)
g. What point is taken to $(10, 3)$?



93. For what value of k is the line $2x + ky = 3$ perpendicular to the line $4x + y = 1$? For what value of k are the lines parallel?

94. Find the line that passes through the point $(1, 2)$ and through the point of intersection of the two lines $x + 2y = 3$ and $2x - 3y = -1$.

95. Midpoint of a line segment Show that the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

is the midpoint of the line segment joining $P(x_1, y_1)$ to $Q(x_2, y_2)$.

96. The distance from a point to a line We can find the distance from a point $P(x_0, y_0)$ to a line $L: Ax + By = C$ by taking the following steps (there is a somewhat faster method in Section 12.5):

1. Find an equation for the line M through P perpendicular to L.
2. Find the coordinates of the point Q in which M and L intersect.
3. Find the distance from P to Q.

Use these steps to find the distance from P to L in each of the following cases.

a. $P(2, 1), \quad L: y = x + 2$

b. $P(4, 6), \quad L: 4x + 3y = 12$

c. $P(a, b), \quad L: x = -1$

d. $P(x_0, y_0), \quad L: Ax + By = C$

1.3 Functions and Their Graphs

Functions are the major objects we deal with in calculus because they are key to describing the real world in mathematical terms. This section reviews the ideas of functions, their graphs, and ways of representing them.

Functions; Domain and Range

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels from an initial location along a straight line path depends on its speed.

In each case, the value of one variable quantity, which we might call y, depends on the value of another variable quantity, which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x. Often the value of y is given by a *rule* or formula that says how to calculate it from the variable x. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r.

In calculus we may want to refer to an unspecified function without having any particular formula in mind. A symbolic way to say "y is a function of x" is by writing

$$y = f(x) \qquad (\text{“}y \text{ equals } f \text{ of } x\text{”})$$

In this notation, the symbol f represents the function. The letter x, called the **independent variable**, represents the input value of f, and y, the **dependent variable**, represents the corresponding output **value** of f at x.

DEFINITION Function

A **function** from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

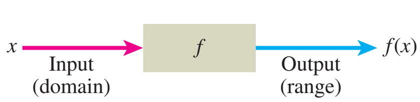


FIGURE 1.22 A diagram showing a function as a kind of machine.

The set D of all possible input values is called the **domain** of the function. The set of all values of $f(x)$ as x varies throughout D is called the **range** of the function. The range may not include every element in the set Y.

The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers. (In Chapters 13–16 many variables may be involved.)

Think of a function f as a kind of machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.22). The function keys on a calculator give an example of a function as a machine. For instance, the $\sqrt{x}$ key on a calculator gives an output value (the square root) whenever you enter a nonnegative number x and press the $\sqrt{x}$ key. The output value appearing in the display is usually a decimal approximation to the square root of x. If you input a number $x < 0$, then the calculator will indicate an error because $x < 0$ is not in the domain of the function and cannot be accepted as an input. The $\sqrt{x}$ key on a calculator is not the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$ because it is limited to decimal outputs and has only finitely many inputs.

A function can also be pictured as an **arrow diagram** (Figure 1.23). Each arrow associates an element of the domain D to a unique or single element in the set Y. In Figure 1.23, the arrows indicate that $f(a)$ is associated with a, $f(x)$ is associated with x, and so on.

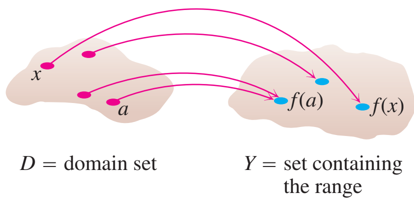


FIGURE 1.23 A function from a set D to a set Y assigns a unique element of Y to each element in D.

The domain of a function may be restricted by context. For example, the domain of the area function given by $A = \pi r^2$ only allows the radius r to be positive. When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values, the so-called **natural domain**. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the function to, say, positive values of x, we would write "$y = x^2, x > 0$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation, the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals. The intervals may be open, closed, or half open, and may be finite or infinite.

EXAMPLE 1 Identifying Domain and Range

Verify the domains and ranges of these functions.

Function	**Domain (x)**	**Range (y)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. *We cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is [0, 1]. ■

Graphs of Functions

Another way to visualize a function is its graph. If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is sketched in Figure 1.24.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above the point x. The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.25).

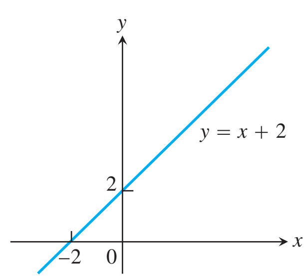


FIGURE 1.24 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

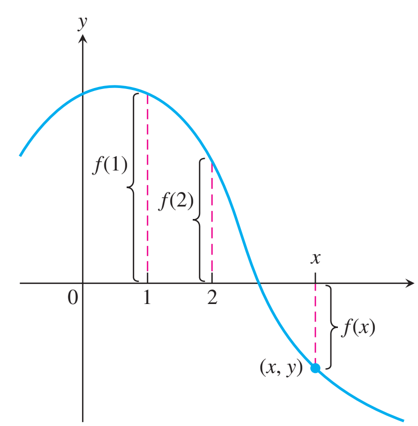


FIGURE 1.25 If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

EXAMPLE 2 Sketching a Graph

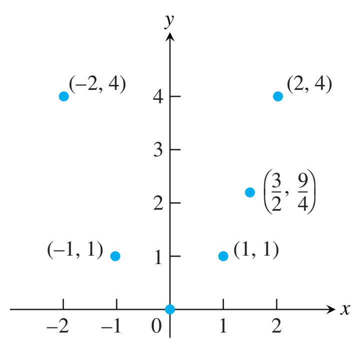
Graph the function $y = x^2$ over the interval $[-2, 2]$.

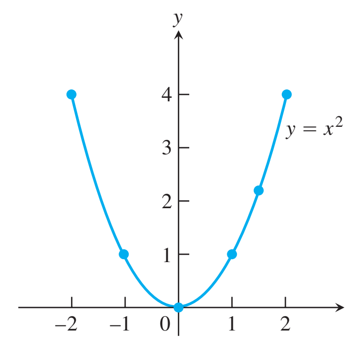
Solution

1. Make a table of xy-pairs that satisfy the function rule, in this case the equation $y = x^2$.

2. Plot the points (x, y) whose coordinates appear in the table. Use fractions when they are convenient computationally.

3. Draw a smooth curve through the plotted points. Label the curve with its equation.





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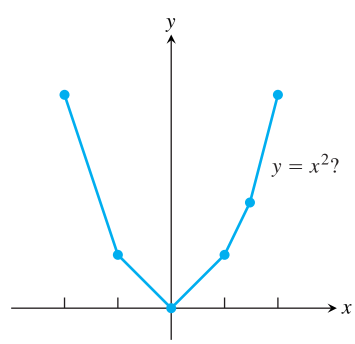
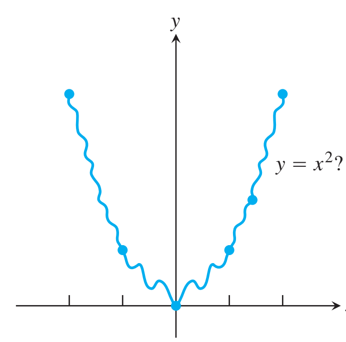
Computers and graphing calculators graph functions in much this way—by stringing together plotted points—and the same question arises.

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?





To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? The answer lies in calculus, as we will see in Chapter 4. There we will use the *derivative* to find a curve's shape between plotted points. Meanwhile we will have to settle for plotting points and connecting them as best we can.

EXAMPLE 3 Evaluating a Function from Its Graph

The graph of a fruit fly population p is shown in Figure 1.26.

(a) Find the populations after 20 and 45 days.

(b) What is the (approximate) range of the population function over the time interval $0 \le t \le 50$?

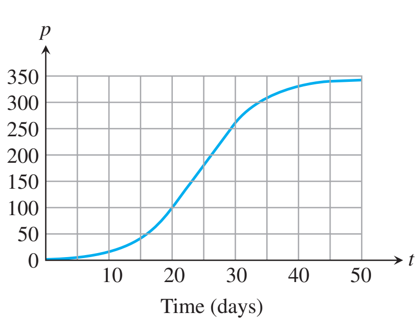


FIGURE 1.26 Graph of a fruit fly population versus time (Example 3).

Solution

(a) We see from Figure 1.26 that the point (20, 100) lies on the graph, so the value of the population p at 20 is $p(20) = 100$. Likewise, $p(45)$ is about 340.

(b) The range of the population function over $0 \le t \le 50$ is approximately [0, 345]. We also observe that the population appears to get closer and closer to the value $p = 350$ as time advances. ■

Representing a Function Numerically

We have seen how a function may be represented algebraically by a formula (the area function) and visually by a graph (Examples 2 and 3). Another way to represent a function is **numerically**, through a table of values. Numerical representations are often used by engineers and applied scientists. From an appropriate table of values, a graph of the function can be obtained using the method illustrated in Example 2, possibly with the aid of a computer. The graph of only the tabled points is called a **scatterplot**.

EXAMPLE 4 A Function Defined by a Table of Values

Musical notes are pressure waves in the air that can be recorded. The data in Table 1.2 give recorded pressure displacement versus time in seconds of a musical note produced by a tuning fork. The table provides a representation of the pressure function over time. If we first make a scatterplot and then connect the data points (t, p) from the table, we obtain the graph shown in Figure 1.27.

TABLE 1.2 Tuning fork data

Time	Pressure	Time	Pressure
0.00091	−0.080	0.00362	0.217
0.00108	0.200	0.00379	0.480
0.00125	0.480	0.00398	0.681
0.00144	0.693	0.00416	0.810
0.00162	0.816	0.00435	0.827
0.00180	0.844	0.00453	0.749
0.00198	0.771	0.00471	0.581
0.00216	0.603	0.00489	0.346
0.00234	0.368	0.00507	0.077
0.00253	0.099	0.00525	−0.164
0.00271	−0.141	0.00543	−0.320
0.00289	−0.309	0.00562	−0.354
0.00307	−0.348	0.00579	−0.248
0.00325	−0.248	0.00598	−0.035
0.00344	−0.041		

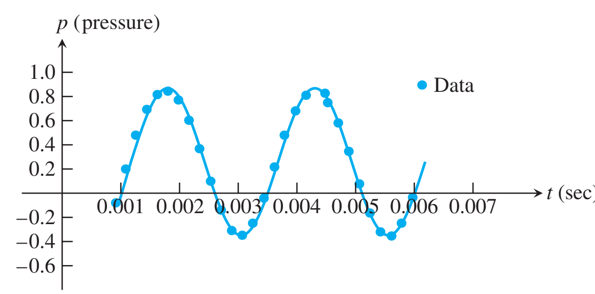


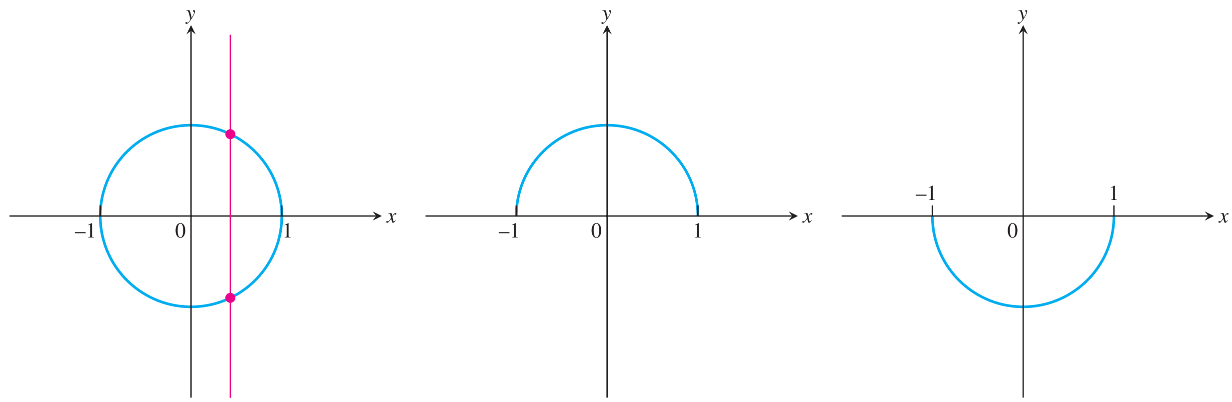
FIGURE 1.27 A smooth curve through the plotted points gives a graph of the pressure function represented by Table 1.2.

■

The Vertical Line Test

Not every curve you draw is the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so no *vertical line* can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function since some vertical lines intersect the circle twice (Figure 1.28a). If a is in the domain of a function f, then the vertical line $x = a$ will intersect the graph of f in the single point $(a, f(a))$.

The circle in Figure 1.28a, however, does contain the graphs of *two* functions of x; the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.28b and 1.28c).



(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

FIGURE 1.28 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

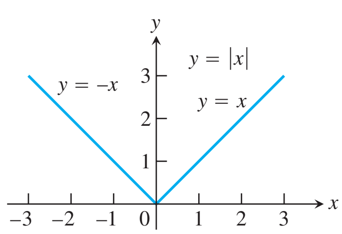


FIGURE 1.29 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

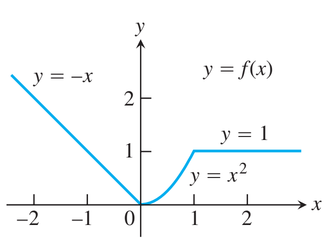


FIGURE 1.30 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 5).

Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Figure 1.29. Here are some other examples.

EXAMPLE 5 Graphing Piecewise-Defined Functions

The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x. The values of f are given by: $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.30). ■

EXAMPLE 6 The Greatest Integer Function

The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$ or int x. Figure 1.31 shows the graph. Observe that

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2.$$

■

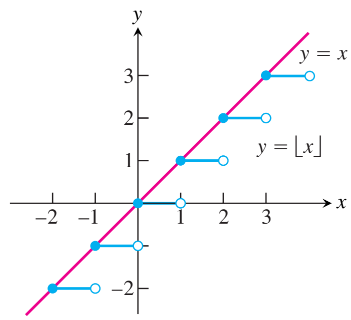


FIGURE 1.31 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 6).

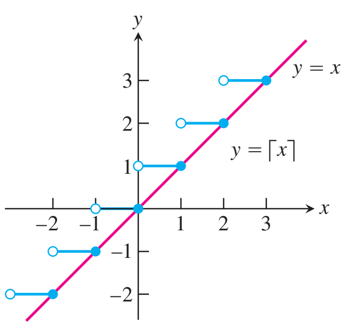


FIGURE 1.32 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 7).

EXAMPLE 7 The Least Integer Function

The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.32 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot which charges $1 for each hour or part of an hour. ■

EXAMPLE 8 Writing Formulas for Piecewise-Defined Functions

Write a formula for the function $y = f(x)$ whose graph consists of the two line segments in Figure 1.33.

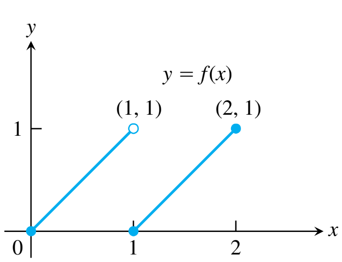


FIGURE 1.33 The segment on the left contains (0, 0) but not (1, 1). The segment on the right contains both of its endpoints (Example 8).

Solution We find formulas for the segments from (0, 0) to (1, 1), and from (1, 0) to (2, 1) and piece them together in the manner of Example 5.

Segment from (0, 0) to (1, 1) The line through (0, 0) and (1, 1) has slope $m = (1 - 0)/(1 - 0) = 1$ and y-intercept $b = 0$. Its slope-intercept equation is $y = x$. The segment from (0, 0) to (1, 1) that includes the point (0, 0) but not the point (1, 1) is the graph of the function $y = x$ restricted to the half-open interval $0 \le x < 1$, namely,

$$y = x, \qquad 0 \le x < 1.$$

Segment from (1, 0) to (2, 1) The line through (1, 0) and (2, 1) has slope $m = (1 - 0)/(2 - 1) = 1$ and passes through the point (1, 0). The corresponding point-slope equation for the line is

$$y = 0 + 1(x - 1), \qquad \text{or} \qquad y = x - 1.$$

The segment from (1, 0) to (2, 1) that includes both endpoints is the graph of $y = x - 1$ restricted to the closed interval $1 \le x \le 2$, namely,

$$y = x - 1, \qquad 1 \le x \le 2.$$

Piecewise formula Combining the formulas for the two pieces of the graph, we obtain

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2. \end{cases}$$ ■

EXERCISES 1.3

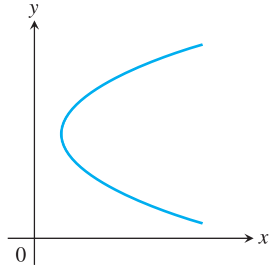
Functions

In Exercises 1–6, find the domain and range of each function.

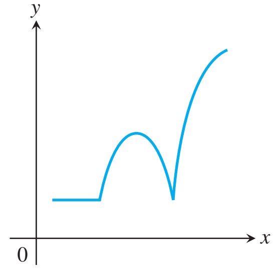
1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(t) = \dfrac{1}{\sqrt{t}}$
4. $F(t) = \dfrac{1}{1 + \sqrt{t}}$
5. $g(z) = \sqrt{4 - z^2}$
6. $g(z) = \dfrac{1}{\sqrt{4 - z^2}}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

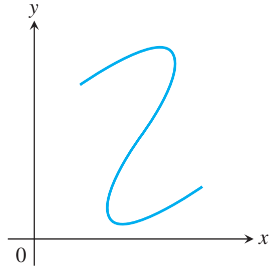
7. a.



b.



8. a.



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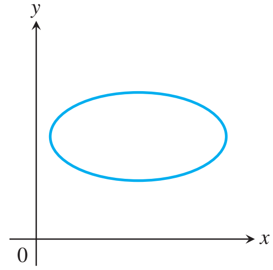


9. Consider the function $y = \sqrt{(1/x) - 1}$.
   a. Can x be negative?
   b. Can $x = 0$?
   c. Can x be greater than 1?
   d. What is the domain of the function?
10. Consider the function $y = \sqrt{2 - \sqrt{x}}$.
   a. Can x be negative?
   b. Can $\sqrt{x}$ be greater than 2?
   c. What is the domain of the function?

Finding Formulas for Functions

11. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.
12. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.
13. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.
14. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.

Functions and Graphs

Find the domain and graph the functions in Exercises 15–20.

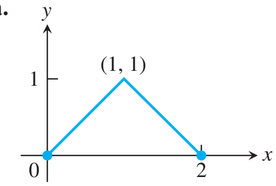
15. $f(x) = 5 - 2x$
16. $f(x) = 1 - 2x - x^2$
17. $g(x) = \sqrt{|x|}$
18. $g(x) = \sqrt{-x}$
19. $F(t) = t/|t|$
20. $G(t) = 1/|t|$
21. Graph the following equations and explain why they are not graphs of functions of x.
   a. $|y| = x$
   b. $y^2 = x^2$
22. Graph the following equations and explain why they are not graphs of functions of x.
   a. $|x| + |y| = 1$
   b. $|x + y| = 1$

Piecewise-Defined Functions

Graph the functions in Exercises 23–26.

23. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$
24. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$
25. $F(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & x > 1 \end{cases}$
26. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$
27. Find a formula for each function graphed.

a.



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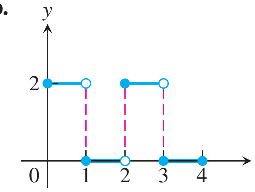
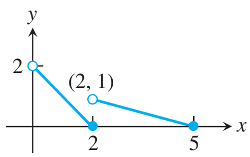
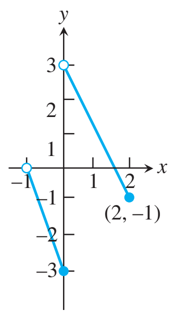


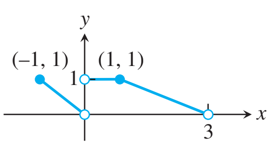
28. a.



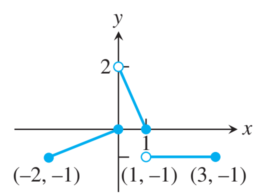
b.



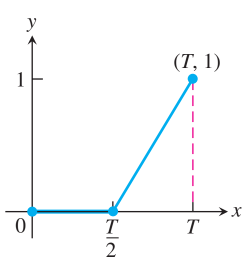
29. a.



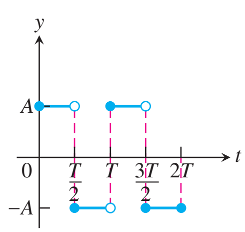
b.



30. a.



b.



T 31. a. Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which
$$\frac{x}{2} > 1 + \frac{4}{x}.$$

b. Confirm your findings in part (a) algebraically.

T 32. a. Graph the functions $f(x) = 3/(x - 1)$ and $g(x) = 2/(x + 1)$ together to identify the values of x for which
$$\frac{3}{x - 1} < \frac{2}{x + 1}.$$

b. Confirm your findings in part (a) algebraically.

The Greatest and Least Integer Functions

33. For what values of x is

a. $\lfloor x \rfloor = 0$? **b.** $\lceil x \rceil = 0$?

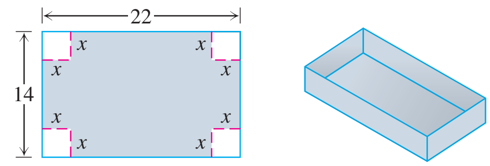
34. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

35. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

36. Graph the function
$$f(x) = \begin{cases} \lfloor x \rfloor, & x \ge 0 \\ \lceil x \rceil, & x < 0 \end{cases}$$
Why is $f(x)$ called the *integer part* of x?

Theory and Examples

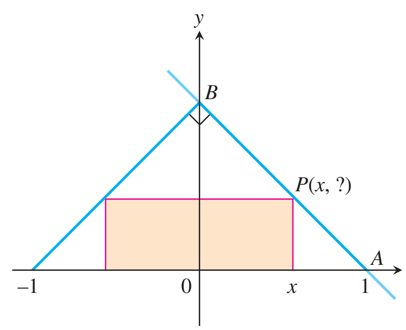
37. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 14 in. by 22 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



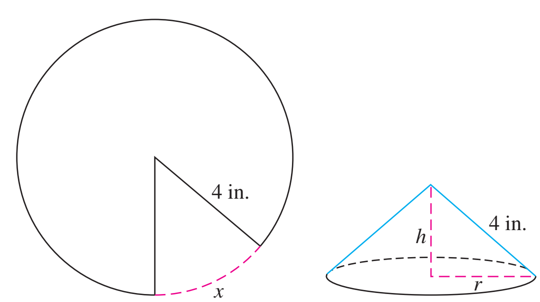
38. The figure shown here shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.

a. Express the y-coordinate of P in terms of x. (You might start by writing an equation for the line AB.)

b. Express the area of the rectangle in terms of x.



39. A cone problem Begin with a circular piece of paper with a 4 in. radius as shown in part (a). Cut out a sector with an arc length of x. Join the two edges of the remaining portion to form a cone with radius r and height h, as shown in part (b).



(a) (b)

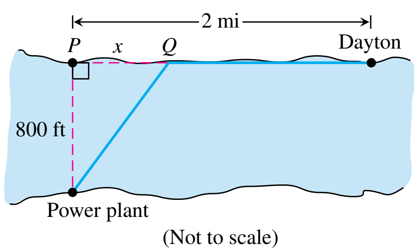
a. Explain why the circumference of the base of the cone is $8\pi - x$.

b. Express the radius r as a function of x.

c. Express the height h as a function of x.

d. Express the volume V of the cone as a function of x.

40. Industrial costs Dayton Power and Light, Inc., has a power plant on the Miami River where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs \$180 per foot across the river and \$100 per foot along the land.



a. Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

b. Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.

41. For a curve to be *symmetric about the x-axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x-axis is not the graph of a function, unless the function is $y = 0$.

42. A magic trick You may have heard of a magic trick that goes like this: Take any number. Add 5. Double the result. Subtract 6. Divide by 2. Subtract 2. Now tell me your answer, and I'll tell you what you started with. Pick a number and try it.

You can see what is going on if you let x be your original number and follow the steps to make a formula $f(x)$ for the number you end up with.

1.4 Identifying Functions; Mathematical Models

There are a number of important types of functions frequently encountered in calculus. We identify and briefly summarize them here.

Linear Functions A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function**. Figure 1.34 shows an array of lines $f(x) = mx$ where $b = 0$, so these lines pass through the origin. Constant functions result when the slope $m = 0$ (Figure 1.35).

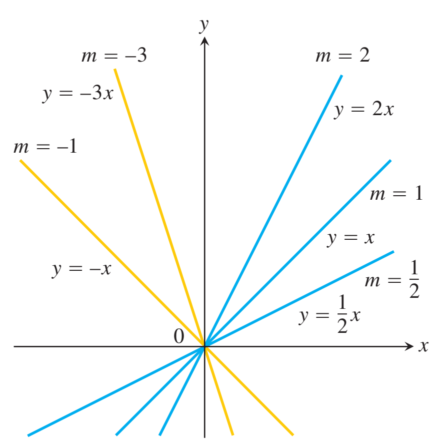


FIGURE 1.34 The collection of lines $y = mx$ has slope m and all lines pass through the origin.

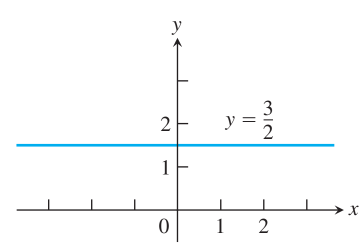


FIGURE 1.35 A constant function has slope $m = 0$.

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.36. These functions are defined for all real values of x. Notice that as the power n gets larger, the curves tend to flatten toward the x-axis on the interval $(-1, 1)$, and also rise more steeply for $|x| > 1$. Each curve passes through the point (1, 1) and through the origin.

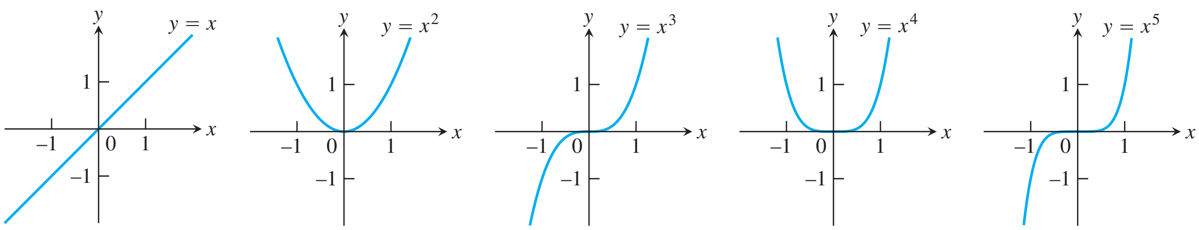


FIGURE 1.36 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$ defined for $-\infty < x < \infty$.

(b) $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 1.37. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$ which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes.

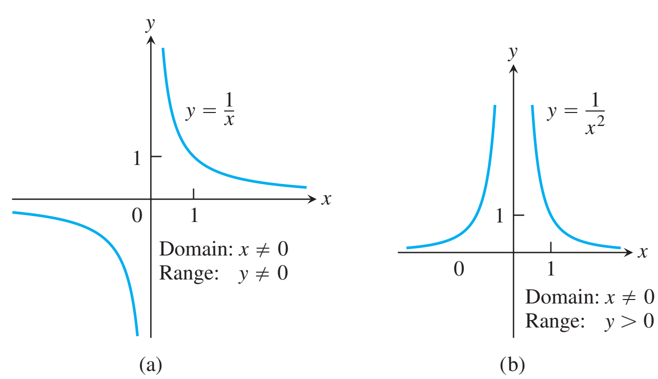


FIGURE 1.37 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.38 along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

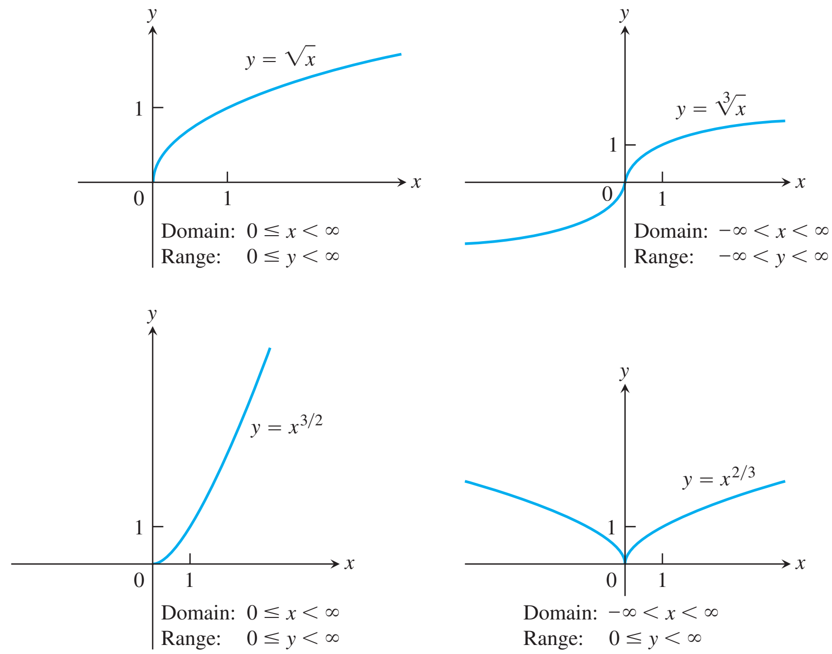


FIGURE 1.38 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.39 shows the graphs of three polynomials. You will learn how to graph polynomials in Chapter 4.

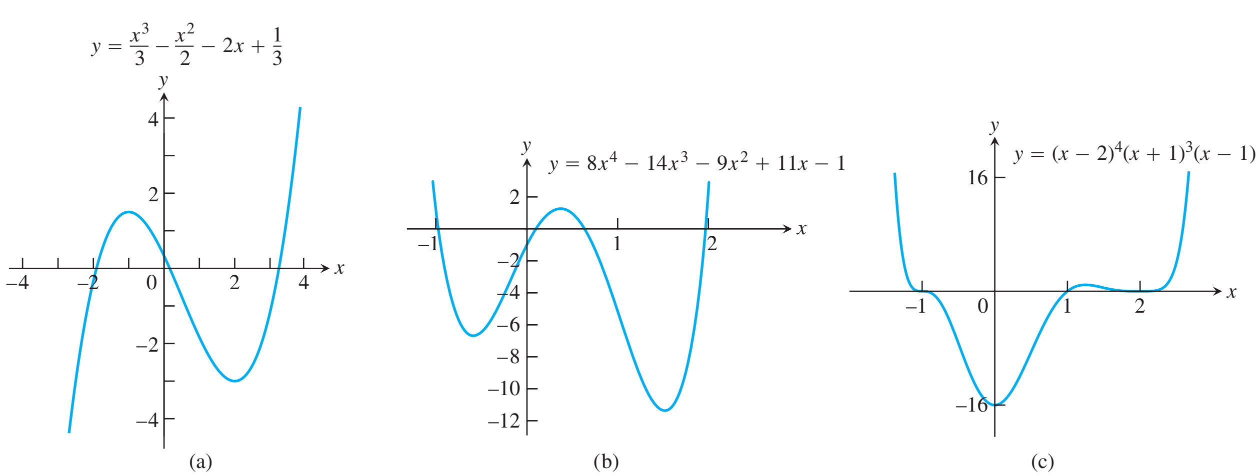


FIGURE 1.39 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. For example, the function

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

is a rational function with domain $\{x \mid x \neq -4/7\}$. Its graph is shown in Figure 1.40a with the graphs of two other rational functions in Figures 1.40b and 1.40c.

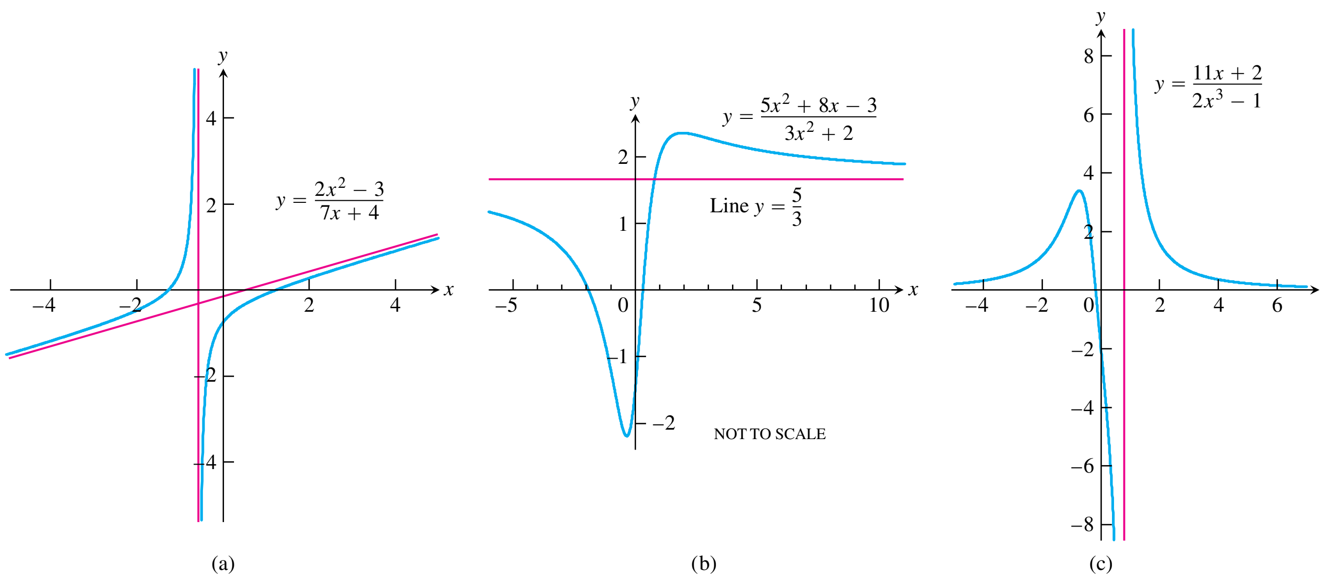


FIGURE 1.40 Graphs of three rational functions.

Algebraic Functions An **algebraic function** is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots). Rational functions are special cases of algebraic functions. Figure 1.41 displays the graphs of three algebraic functions.

Trigonometric Functions We review trigonometric functions in Section 1.6. The graphs of the sine and cosine functions are shown in Figure 1.42.

Exponential Functions Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$. So an exponential function never assumes the value 0. The graphs of some exponential functions are shown in Figure 1.43. The calculus of exponential functions is studied in Chapter 7.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions, and the

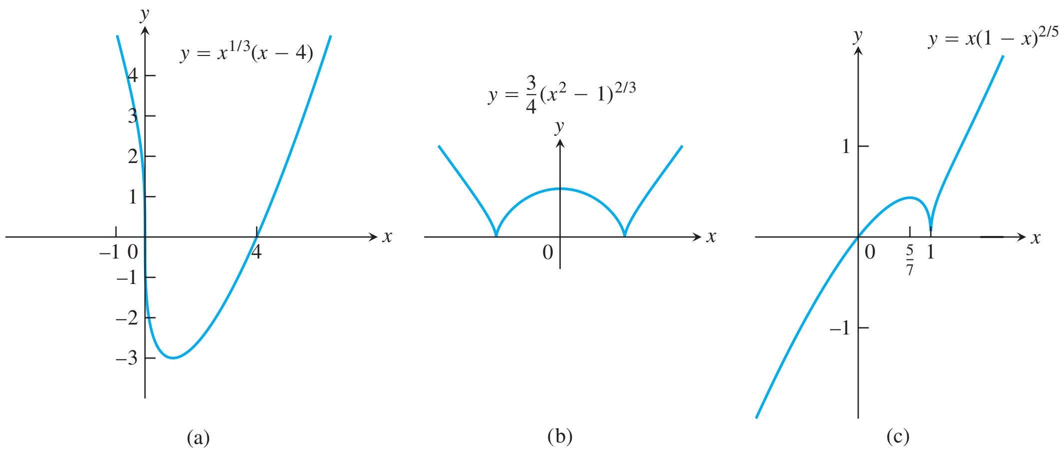


FIGURE 1.41 Graphs of three algebraic functions.

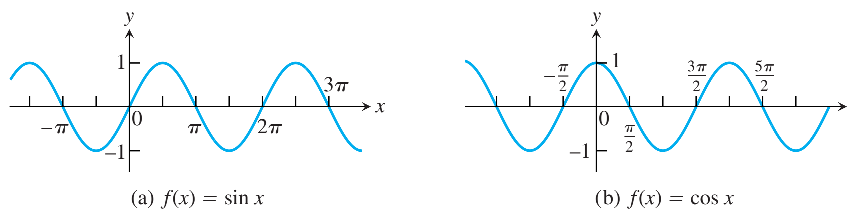


FIGURE 1.42 Graphs of the sine and cosine functions.

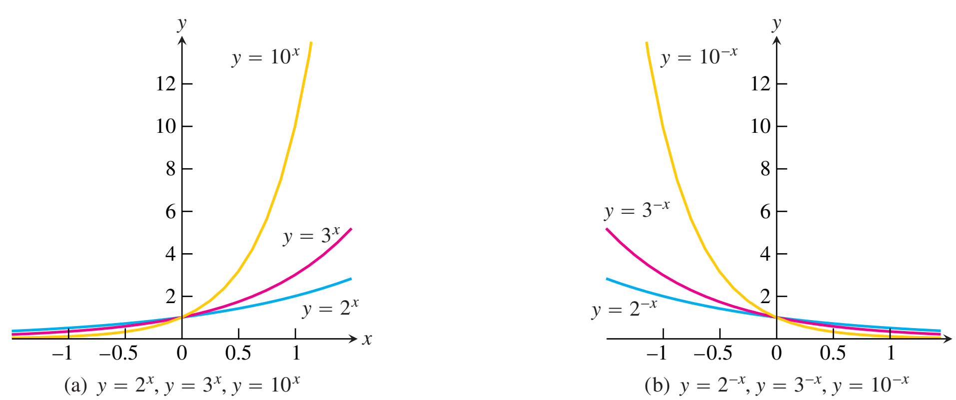


FIGURE 1.43 Graphs of exponential functions.

calculus of these functions is studied in Chapter 7. Figure 1.44 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many

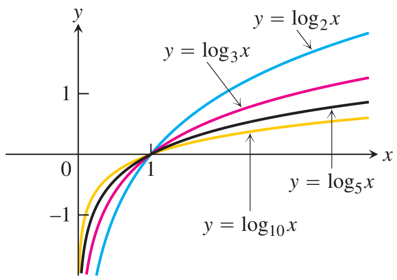


FIGURE 1.44 Graphs of four logarithmic functions.

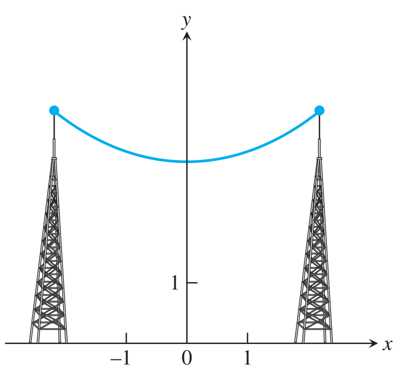


FIGURE 1.45 Graph of a catenary or hanging cable. (The Latin word *catena* means "chain.")

other functions as well (such as the hyperbolic functions studied in Chapter 7). An example of a transcendental function is a **catenary**. Its graph takes the shape of a cable, like a telephone line or TV cable, strung from one support to another and hanging freely under its own weight (Figure 1.45).

EXAMPLE 1 Recognizing Functions

Identify each function given here as one of the types of functions we have discussed. Keep in mind that some functions can fall into more than one category. For example, $f(x) = x^2$ is both a power function and a polynomial of second degree.

(a) $f(x) = 1 + x - \frac{1}{2}x^5$ **(b)** $g(x) = 7^x$ **(c)** $h(z) = z^7$

(d) $y(t) = \sin\left(t - \frac{\pi}{4}\right)$

Solution

(a) $f(x) = 1 + x - \frac{1}{2}x^5$ is a polynomial of degree 5.

(b) $g(x) = 7^x$ is an exponential function with base 7. Notice that the variable x is the exponent.

(c) $h(z) = z^7$ is a power function. (The variable z is the base.)

(d) $y(t) = \sin\left(t - \frac{\pi}{4}\right)$ is a trigonometric function. ■

Increasing Versus Decreasing Functions

If the graph of a function *climbs* or *rises* as you move from left to right, we say that the function is *increasing*. If the graph *descends* or *falls* as you move from left to right, the function is *decreasing*. We give formal definitions of increasing functions and decreasing functions in Section 4.3. In that section, you will learn how to find the intervals over which a function is increasing and the intervals where it is decreasing. Here are examples from Figures 1.36, 1.37, and 1.38.

Function	Where increasing	Where decreasing
$y = x^2$	$0 \le x < \infty$	$-\infty < x \le 0$
$y = x^3$	$-\infty < x < \infty$	Nowhere
$y = 1/x$	Nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = \sqrt{x}$	$0 \le x < \infty$	Nowhere
$y = x^{2/3}$	$0 \le x < \infty$	$-\infty < x \le 0$

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

DEFINITIONS Even Function, Odd Function

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

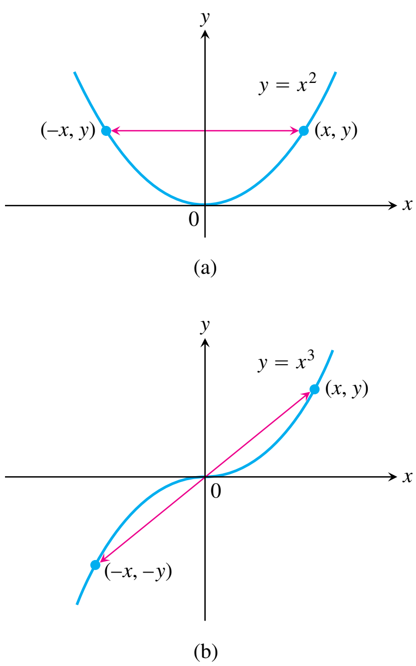


FIGURE 1.46 In part (a) the graph of $y = x^2$ (an even function) is symmetric about the y-axis. The graph of $y = x^3$ (an odd function) in part (b) is symmetric about the origin.

The names even and odd come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x (because $(-x)^2 = x^2$ and $(-x)^4 = x^4$). If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x (because $(-x)^1 = -x$ and $(-x)^3 = -x^3$).

The graph of an even function is **symmetric about the y-axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.46a). A reflection across the y-axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.46b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply both x and $-x$ must be in the domain of f.

EXAMPLE 2 Recognizing Even and Odd Functions

$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.47a).

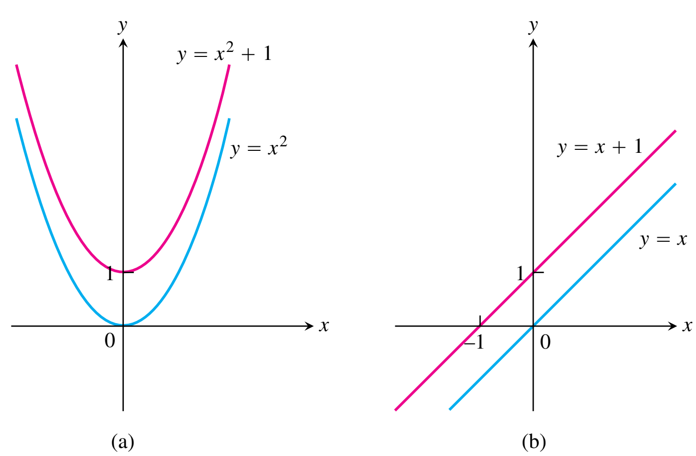


FIGURE 1.47 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost (Example 2).

$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.
	Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.47b). ■

Mathematical Models

To help us better understand our world, we often describe a particular phenomenon mathematically (by means of a function or an equation, for instance). Such a **mathematical model** is an idealization of the real-world phenomenon and is seldom a completely accurate representation. Although any model has its limitations, a good one can provide valuable results and conclusions. A model allows us to reach conclusions, as illustrated in Figure 1.48.

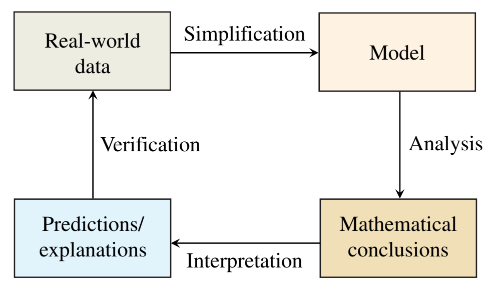


FIGURE 1.48 A flow of the modeling process beginning with an examination of real-world data.

Most models simplify reality and can only *approximate* real-world behavior. One simplifying relationship is *proportionality*.

> **DEFINITION Proportionality**
>
> Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if
>
> $$y = kx$$
>
> for some nonzero constant k.

The definition means that the graph of y versus x lies along a straight line through the origin. This graphical observation is useful in testing whether a given data collection reasonably assumes a proportionality relationship. If a proportionality is reasonable, a plot of one variable against the other should approximate a straight line through the origin.

EXAMPLE 3 Kepler's Third Law

A famous proportionality, postulated by the German astronomer Johannes Kepler in the early seventeenth century, is his third law. If T is the period in days for a planet to complete one full orbit around the sun, and R is the mean distance of the planet to the sun, then Kepler postulated that T is proportional to R raised to the 3/2 power. That is, for some constant k,

$$T = kR^{3/2}.$$

Let's compare his law to the data in Table 1.3 taken from the *1993 World Almanac*.

TABLE 1.3 Orbital periods and mean distances of planets from the sun

Planet	T Period (days)	R Mean distance (millions of miles)
Mercury	88.0	36
Venus	224.7	67.25
Earth	365.3	93
Mars	687.0	141.75
Jupiter	4,331.8	483.80
Saturn	10,760.0	887.97
Uranus	30,684.0	1,764.50
Neptune	60,188.3	2,791.05
Pluto	90,466.8	3,653.90

The graphing principle in this example may be new to you. To plot T versus $R^{3/2}$ we first calculate the value of $R^{3/2}$ for each value in Table 1.3. For example, $3653.90^{3/2} \approx 220{,}869.1$ and $36^{3/2} = 216$. The horizontal axis represents $R^{3/2}$ (not R values) and we plot the ordered pairs $(R^{3/2}, T)$ in the coordinate system in Figure 1.49. This plot of ordered pairs or scatterplot gives a graph of the period versus the mean distance to the 3/2 power. We observe that the scatterplot in the figure does lie approximately along a straight line that projects through the origin. By picking two points that lie on that line we can easily estimate the slope, which is the constant of proportionality (in days per miles $\times 10^{-4}$).

$$k = slope = \frac{90,466.8 - 88}{220{,}869.1 - 216} \approx 0.410$$

We estimate the model of Kepler's third law to be $T = 0.410R^{3/2}$ (which depends on our choice of units). We need to be careful to point out that this is *not a proof* of Kepler's third

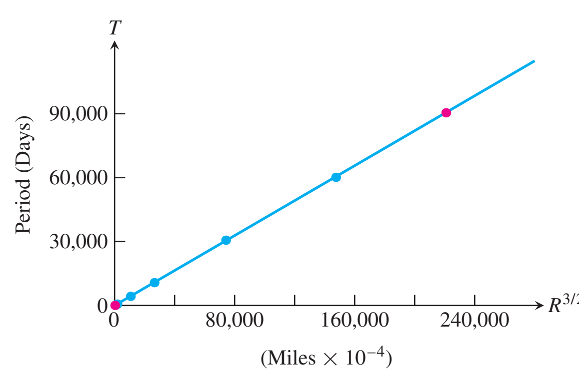


FIGURE 1.49 Graph of Kepler's third law as a proportionality: $T = 0.410R^{3/2}$ (Example 3).

law. We cannot prove or verify a theorem by just looking at some examples. Nevertheless, Figure 1.49 suggests that Kepler's third law is reasonable. ■

The concept of proportionality is one way to test the reasonableness of a conjectured relationship between two variables, as in Example 3. It can also provide the basis for an **empirical model** which comes entirely from a table of collected data.

EXERCISES 1.4

Recognizing Functions

In Exercises 1–4, identify each function as a constant function, linear function, power function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function. Remember that some functions can fall into more than one category.

1. **a.** $f(x) = 7 - 3x$ **b.** $g(x) = \sqrt[5]{x}$
c. $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$ **d.** $r(x) = 8^x$

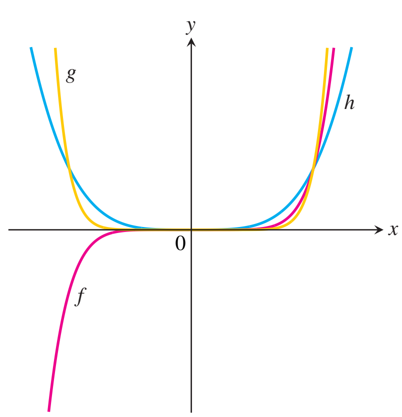
2. **a.** $F(t) = t^4 - t$ **b.** $G(t) = 5^t$
c. $H(z) = \sqrt{z^3 + 1}$ **d.** $R(z) = \sqrt[3]{z^7}$

3. **a.** $y = \dfrac{3 + 2x}{x - 1}$ **b.** $y = x^{5/2} - 2x + 1$
c. $y = \tan \pi x$ **d.** $y = \log_7 x$

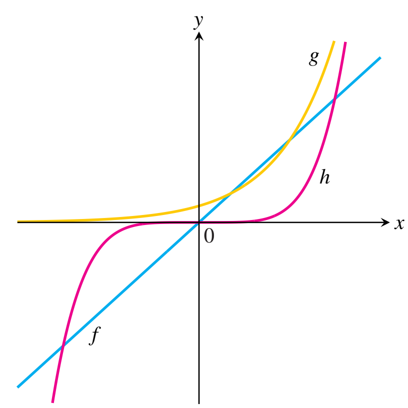
4. **a.** $y = \log_5\left(\dfrac{1}{t}\right)$ **b.** $f(z) = \dfrac{z^5}{\sqrt{z} + 1}$
c. $g(x) = 2^{1/x}$ **d.** $w = 5\cos\left(\dfrac{t}{2} + \dfrac{\pi}{6}\right)$

In Exercises 5 and 6, match each equation with its graph. Do not use a graphing device, and give reasons for your answer.

5. **a.** $y = x^4$ **b.** $y = x^7$ **c.** $y = x^{10}$



6. **a.** $y = 5x$ **b.** $y = 5^x$ **c.** $y = x^5$



Increasing and Decreasing Functions

Graph the functions in Exercises 7–18. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

7. $y = -x^3$
8. $y = -\dfrac{1}{x^2}$
9. $y = -\dfrac{1}{x}$
10. $y = \dfrac{1}{|x|}$
11. $y = \sqrt{|x|}$
12. $y = \sqrt{-x}$
13. $y = x^3/8$
14. $y = -4\sqrt{x}$
15. $y = -x^{3/2}$
16. $y = (-x)^{3/2}$
17. $y = (-x)^{2/3}$
18. $y = -x^{2/3}$

Even and Odd Functions

In Exercises 19–30, say whether the function is even, odd, or neither. Give reasons for your answer.

19. $f(x) = 3$
20. $f(x) = x^{-5}$
21. $f(x) = x^2 + 1$
22. $f(x) = x^2 + x$
23. $g(x) = x^3 + x$
24. $g(x) = x^4 + 3x^2 - 1$

25. $g(x) = \dfrac{1}{x^2 - 1}$

26. $g(x) = \dfrac{x}{x^2 - 1}$

27. $h(t) = \dfrac{1}{t - 1}$

28. $h(t) = |t^3|$

29. $h(t) = 2t + 1$

30. $h(t) = 2|t| + 1$

Proportionality

In Exercises 31 and 32, assess whether the given data sets reasonably support the stated proportionality assumption. Graph an appropriate scatterplot for your investigation and, if the proportionality assumption seems reasonable, estimate the constant of proportionality.

31. **a.** y is proportional to x

y	1	2	3	4	5	6	7	8
x	5.9	12.1	17.9	23.9	29.9	36.2	41.8	48.2

b. y is proportional to $x^{1/2}$

y	3.5	5	6	7	8
x	3	6	9	12	15

32. **a.** y is proportional to 3^x

y	5	15	45	135	405	1215	3645	10,935
x	0	1	2	3	4	5	6	7

b. y is proportional to $\ln x$

y	2	4.8	5.3	6.5	8.0	10.5	14.4	15.0
x	2.0	5.0	6.0	9.0	14.0	35.0	120.0	150.0

T **33.** The accompanying table shows the distance a car travels during the time the driver is reacting before applying the brakes, and the distance the car travels after the brakes are applied. The distances (in feet) depend on the speed of the car (in miles per hour). Test the reasonableness of the following proportionality assumptions and estimate the constants of proportionality.

a. reaction distance is proportional to speed.

b. braking distance is proportional to the square of the speed.

Speed (mph)	20	25	30	35	40	45	50	55	60	65	70	75	80
Reaction distance (ft)	22	28	33	39	44	50	55	61	66	72	77	83	88
Braking distance (ft)	20	28	41	53	72	93	118	149	182	221	266	318	376

34. In October 2002, astronomers discovered a rocky, icy mini-planet tentatively named "Quaoar" circling the sun far beyond Neptune. The new planet is about 4 billion miles from Earth in an outer fringe of the solar system known as the Kuiper Belt. Using Kepler's third law, estimate the time T it takes Quaoar to complete one full orbit around the sun.

T **35.** **Spring elongation** The response of a spring to various loads must be modeled to design a vehicle such as a dump truck, utility vehicle, or a luxury car that responds to road conditions in a desired way. We conducted an experiment to measure the stretch y of a spring in inches as a function of the number x of units of mass placed on the spring.

x (number of units of mass)	0	1	2	3	4	5
y (elongation in inches)	0	0.875	1.721	2.641	3.531	4.391

x (number of units of mass)	6	7	8	9	10
y (elongation in inches)	5.241	6.120	6.992	7.869	8.741

a. Make a scatterplot of the data to test the reasonableness of the hypothesis that stretch y is proportional to the mass x.

b. Estimate the constant of proportionality from your graph obtained in part (a).

c. Predict the elongation of the spring for 13 units of mass.

36. **Ponderosa pines** In the table, x represents the girth (distance around) of a pine tree measured in inches (in.) at shoulder height; y represents the board feet (bf) of lumber finally obtained.

x (in.)	17	19	20	23	25	28	32	38	39	41
y (bf)	19	25	32	57	71	113	123	252	259	294

Formulate and test the following two models: that usable board feet is proportional to **(a)** the square of the girth and **(b)** the cube of the girth. Does one model provide a better "explanation" than the other?

1.5 Combining Functions; Shifting and Scaling Graphs

In this section we look at the main ways functions are combined or transformed to form new functions.

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$(f + g)(x) = f(x) + g(x).$$
$$(f - g)(x) = f(x) - g(x).$$
$$(fg)(x) = f(x)g(x).$$

Notice that the + sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the + on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 1 Combining Functions Algebraically

The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x},$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

■

The graph of the function $f + g$ is obtained from the graphs of f and g by adding the corresponding y-coordinates $f(x)$ and $g(x)$ at each point $x \in D(f) \cap D(g)$, as in Figure 1.50. The graphs of $f + g$ and $f \cdot g$ from Example 1 are shown in Figure 1.51.

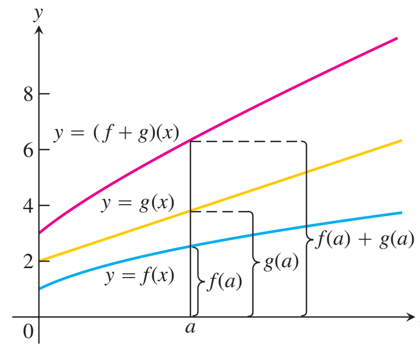


FIGURE 1.50 Graphical addition of two functions.

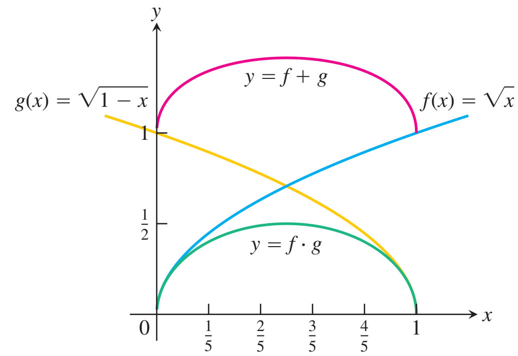


FIGURE 1.51 The domain of the function $f + g$ is the intersection of the domains of f and g, the interval $[0, 1]$ on the x-axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

Composite Functions

Composition is another method for combining functions.

> **DEFINITION Composition of Functions**
>
> If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by
>
> $$(f \circ g)(x) = f(g(x)).$$
>
> The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

The definition says that $f \circ g$ can be formed when the range of g lies in the domain of f. To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$. Figure 1.52 pictures $f \circ g$ as a machine diagram and Figure 1.53 shows the composite as an arrow diagram.

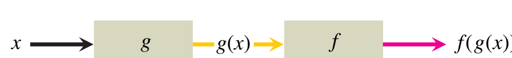


FIGURE 1.52 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

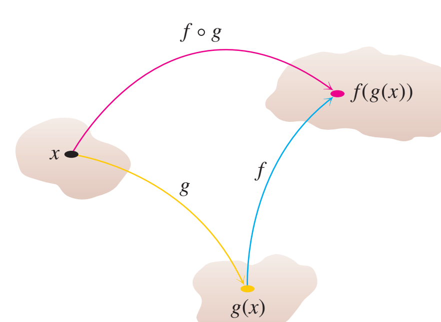


FIGURE 1.53 Arrow diagram for $f \circ g$.

EXAMPLE 2 Viewing a Function as a Composite

The function $y = \sqrt{1 - x^2}$ can be thought of as first calculating $1 - x^2$ and then taking the square root of the result. The function y is the composite of the function $g(x) = 1 - x^2$ and the function $f(x) = \sqrt{x}$. Notice that $1 - x^2$ cannot be negative. The domain of the composite is $[-1, 1]$. ■

To evaluate the composite function $g \circ f$ (when defined), we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 3 Finding Formulas for Composites

If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

	Composite	Domain
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = \left(\sqrt{x}\right)^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$.

Shifting a Graph of a Function

To shift the graph of a function $y = f(x)$ straight up, add a positive constant to the right-hand side of the formula $y = f(x)$.

To shift the graph of a function $y = f(x)$ straight down, add a negative constant to the right-hand side of the formula $y = f(x)$.

To shift the graph of $y = f(x)$ to the left, add a positive constant to x. To shift the graph of $y = f(x)$ to the right, add a negative constant to x.

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\lvert k \rvert$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\lvert h \rvert$ units if $h < 0$

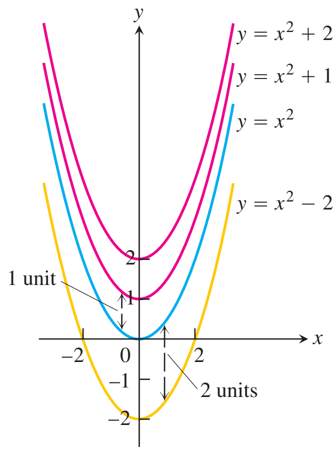


FIGURE 1.54 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Example 4a and b).

EXAMPLE 4 Shifting a Graph

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.54).

(b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.54).

(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Figure 1.55).

(d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.56).

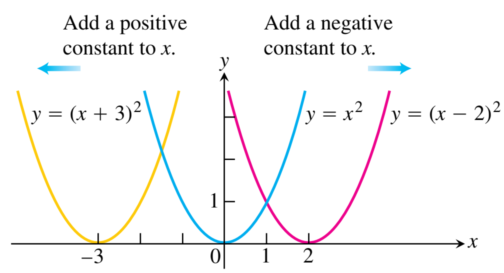


FIGURE 1.55 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x (Example 4c).

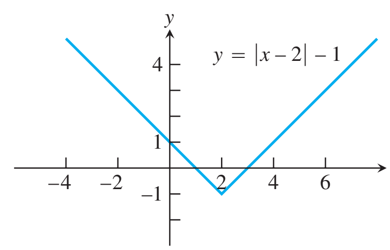


FIGURE 1.56 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 4d).

Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = -1$.

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$,

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

EXAMPLE 5 Scaling and Reflecting a Graph

(a) **Vertical:** Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by 1/3 compresses the graph by a factor of 3 (Figure 1.57).

(b) **Horizontal:** The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 (Figure 1.58). Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

(c) **Reflection:** The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = \sqrt{-x}$ is a reflection across the y-axis (Figure 1.59).

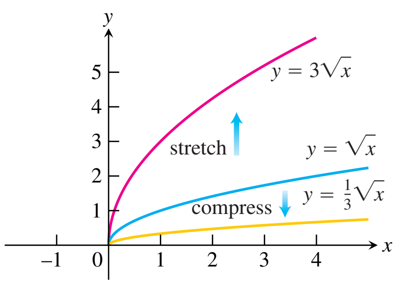


FIGURE 1.57 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5a).

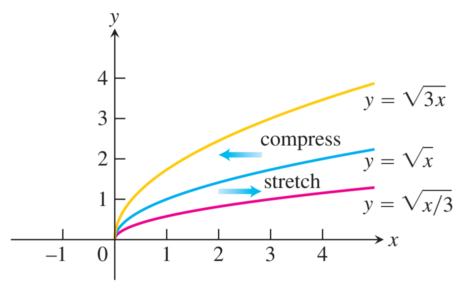


FIGURE 1.58 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5b).

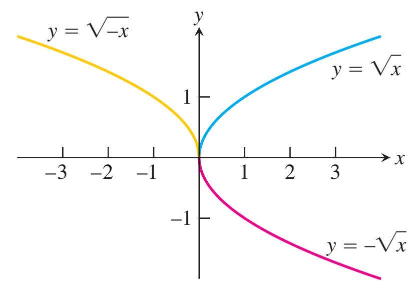


FIGURE 1.59 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 5c). ■

EXAMPLE 6 Combining Scalings and Reflections

Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure 1.60a), find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis (Figure 1.60b).

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis (Figure 1.60c).

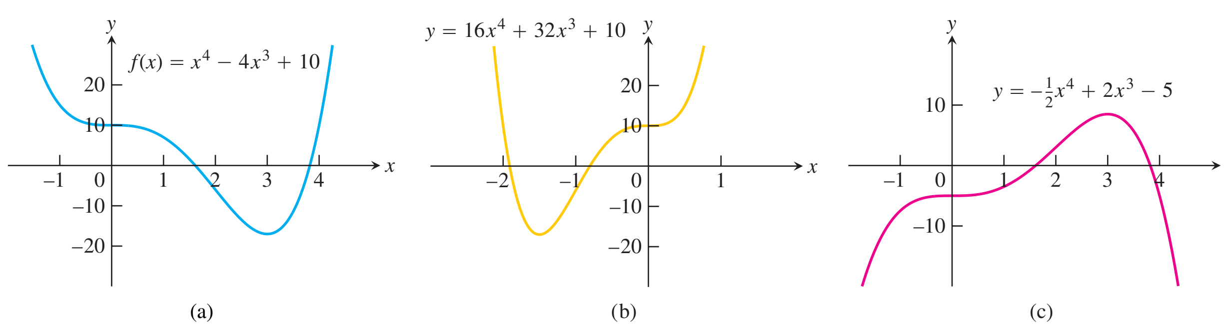


FIGURE 1.60 (a) The original graph of f. (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y-axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x-axis (Example 6).

Solution

(a) The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f

$$y = f(-2x) = (-2x)^4 - 4(-2x)^3 + 10$$
$$= 16x^4 + 32x^3 + 10.$$

(b) The formula is

$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5.$$

Ellipses

Substituting cx for x in the standard equation for a circle of radius r centered at the origin gives

$$c^2x^2 + y^2 = r^2. \tag{1}$$

If $0 < c < 1$, the graph of Equation (1) horizontally stretches the circle; if $c > 1$ the circle is compressed horizontally. In either case, the graph of Equation (1) is an ellipse (Figure 1.61). Notice in Figure 1.61 that the y-intercepts of all three graphs are always $-r$ and r. In Figure 1.61b, the line segment joining the points $(\pm r/c, 0)$ is called the **major axis** of the ellipse; the **minor axis** is the line segment joining $(0, \pm r)$. The axes of the ellipse are reversed in Figure 1.61c: the major axis is the line segment joining the points $(0, \pm r)$ and the minor axis is the line segment joining the points $(\pm r/c, 0)$. In both cases, the major axis is the line segment having the longer length.

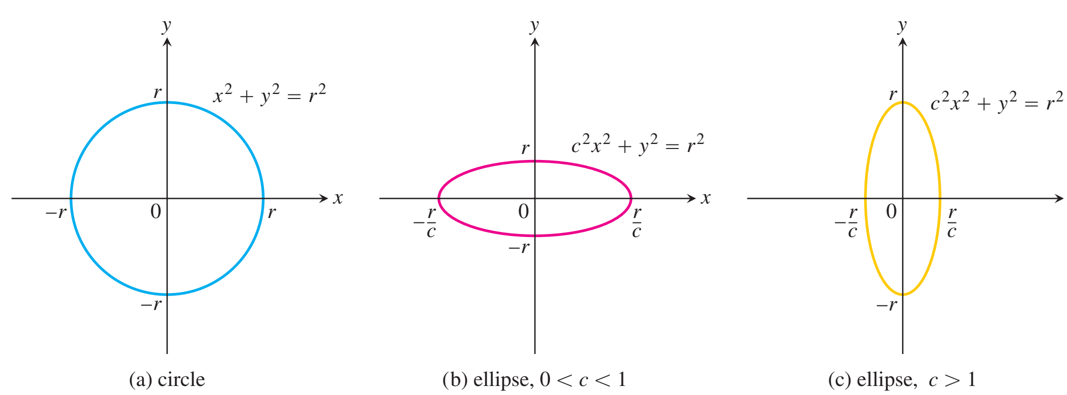


FIGURE 1.61 Horizontal stretchings or compressions of a circle produce graphs of ellipses.

If we divide both sides of Equation (1) by r^2, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2}$$

where $a = r/c$ and $b = r$. If $a > b$, the major axis is horizontal; if $a < b$, the major axis is vertical. The **center** of the ellipse given by Equation (2) is the origin (Figure 1.62).

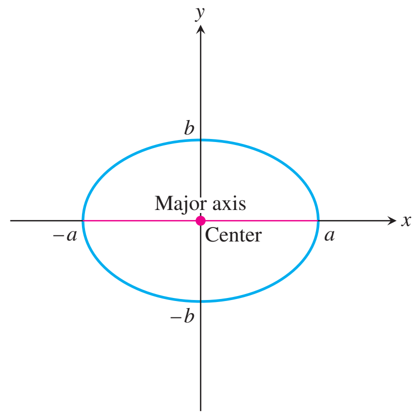


FIGURE 1.62 Graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, where the major axis is horizontal.

Substituting $x - h$ for x, and $y - k$ for y, in Equation (2) results in

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \tag{3}$$

Equation (3) is the **standard equation of an ellipse** with center at (h, k). The geometric definition and properties of ellipses are reviewed in Section 10.1.

EXERCISES 1.5

Sums, Differences, Products, and Quotients

In Exercises 1 and 2, find the domains and ranges of f, g, $f + g$, and $f \cdot g$.

1. $f(x) = x, \quad g(x) = \sqrt{x - 1}$

2. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

In Exercises 3 and 4, find the domains and ranges of f, g, f/g, and g/f.

3. $f(x) = 2, \quad g(x) = x^2 + 1$

4. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

Composites of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a. $f(g(0))$ **b.** $g(f(0))$
c. $f(g(x))$ **d.** $g(f(x))$
e. $f(f(-5))$ **f.** $g(g(2))$
g. $f(f(x))$ **h.** $g(g(x))$

6. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

a. $f(g(1/2))$ **b.** $g(f(1/2))$
c. $f(g(x))$ **d.** $g(f(x))$
e. $f(f(2))$ **f.** $g(g(2))$
g. $f(f(x))$ **h.** $g(g(x))$

7. If $u(x) = 4x - 5$, $v(x) = x^2$, and $f(x) = 1/x$, find formulas for the following.

a. $u(v(f(x)))$ **b.** $u(f(v(x)))$
c. $v(u(f(x)))$ **d.** $v(f(u(x)))$
e. $f(u(v(x)))$ **f.** $f(v(u(x)))$

8. If $f(x) = \sqrt{x}$, $g(x) = x/4$, and $h(x) = 4x - 8$, find formulas for the following.

a. $h(g(f(x)))$ **b.** $h(f(g(x)))$
c. $g(h(f(x)))$ **d.** $g(f(h(x)))$
e. $f(g(h(x)))$ **f.** $f(h(g(x)))$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 9 and 10 as a composite involving one or more of f, g, h, and j.

9. a. $y = \sqrt{x} - 3$ **b.** $y = 2\sqrt{x}$
c. $y = x^{1/4}$ **d.** $y = 4x$
e. $y = \sqrt{(x - 3)^3}$ **f.** $y = (2x - 6)^3$

10. a. $y = 2x - 3$ **b.** $y = x^{3/2}$
c. $y = x^9$ **d.** $y = x - 6$
e. $y = 2\sqrt{x - 3}$ **f.** $y = \sqrt{x^3 - 3}$

11. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$x - 7$	$\sqrt{x}$	
b.	$x + 2$	$3x$	
c.		$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d.	$\dfrac{x}{x - 1}$	$\dfrac{x}{x - 1}$	
e.		$1 + \dfrac{1}{x}$	x
f.	$\dfrac{1}{x}$		x

12. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$\dfrac{1}{x - 1}$	$\lvert x \rvert$	?
b.	?	$\dfrac{x - 1}{x}$	$\dfrac{x}{x + 1}$
c.	?	$\sqrt{x}$	$\lvert x \rvert$
d.	$\sqrt{x}$	?	$\lvert x \rvert$

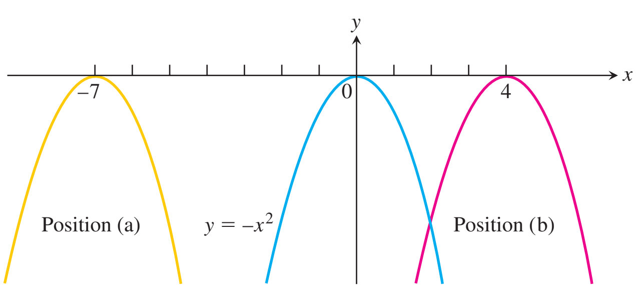
In Exercises 13 and 14, **(a)** write a formula for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

13. $f(x) = \sqrt{x + 1}, \quad g(x) = \dfrac{1}{x}$

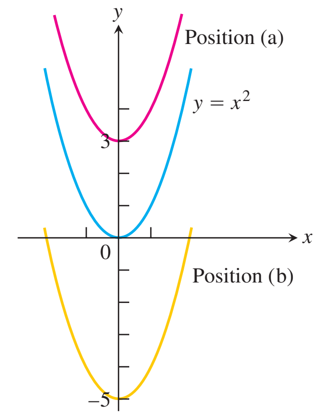
14. $f(x) = x^2, \quad g(x) = 1 - \sqrt{x}$

Shifting Graphs

15. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



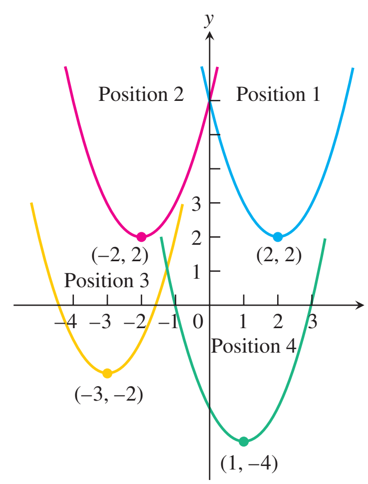
16. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



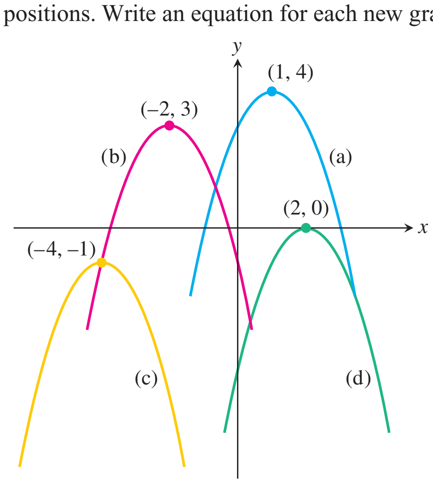
17. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x - 1)^2 - 4$ **b.** $y = (x - 2)^2 + 2$

c. $y = (x + 2)^2 + 2$ **d.** $y = (x + 3)^2 - 2$



18. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



Exercises 19–28 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

19. $x^2 + y^2 = 49$ Down 3, left 2

20. $x^2 + y^2 = 25$ Up 3, left 4

21. $y = x^3$ Left 1, down 1

22. $y = x^{2/3}$ Right 1, down 1

23. $y = \sqrt{x}$ Left 0.81

24. $y = -\sqrt{x}$ Right 3

25. $y = 2x - 7$ Up 7

26. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

27. $y = 1/x$ Up 1, right 1

28. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 29–48.

29. $y = \sqrt{x + 4}$

30. $y = \sqrt{9 - x}$

31. $y = |x - 2|$

32. $y = |1 - x| - 1$

33. $y = 1 + \sqrt{x - 1}$

34. $y = 1 - \sqrt{x}$

35. $y = (x + 1)^{2/3}$

36. $y = (x - 8)^{2/3}$

37. $y = 1 - x^{2/3}$

38. $y + 4 = x^{2/3}$

39. $y = \sqrt[3]{x - 1} - 1$

40. $y = (x + 2)^{3/2} + 1$

41. $y = \dfrac{1}{x - 2}$

42. $y = \dfrac{1}{x} - 2$

43. $y = \dfrac{1}{x} + 2$

44. $y = \dfrac{1}{x + 2}$

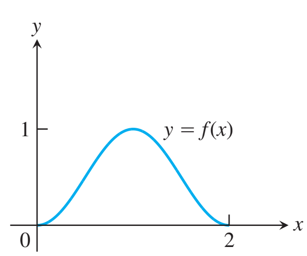
45. $y = \dfrac{1}{(x - 1)^2}$

46. $y = \dfrac{1}{x^2} - 1$

47. $y = \dfrac{1}{x^2} + 1$

48. $y = \dfrac{1}{(x + 1)^2}$

49. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



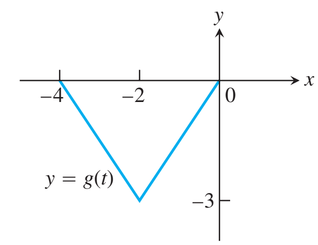
a. $f(x) + 2$ **b.** $f(x) - 1$

c. $2f(x)$ **d.** $-f(x)$

e. $f(x + 2)$ **f.** $f(x - 1)$

g. $f(-x)$ **h.** $-f(x + 1) + 1$

50. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



a. $g(-t)$ **b.** $-g(t)$

c. $g(t) + 3$ **d.** $1 - g(t)$

e. $g(-t + 2)$ **f.** $g(t - 2)$

g. $g(1 - t)$ **h.** $-g(t - 4)$

Vertical and Horizontal Scaling

Exercises 51–60 tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

51. $y = x^2 - 1$, stretched vertically by a factor of 3

52. $y = x^2 - 1$, compressed horizontally by a factor of 2

53. $y = 1 + \dfrac{1}{x^2}$, compressed vertically by a factor of 2

54. $y = 1 + \dfrac{1}{x^2}$, stretched horizontally by a factor of 3

55. $y = \sqrt{x + 1}$, compressed horizontally by a factor of 4

56. $y = \sqrt{x + 1}$, stretched vertically by a factor of 3

57. $y = \sqrt{4 - x^2}$, stretched horizontally by a factor of 2

58. $y = \sqrt{4 - x^2}$, compressed vertically by a factor of 3

59. $y = 1 - x^3$, compressed horizontally by a factor of 3

60. $y = 1 - x^3$, stretched horizontally by a factor of 2

Graphing

In Exercises 61–68, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.36–1.38, and applying an appropriate transformation.

61. $y = -\sqrt{2x + 1}$

62. $y = \sqrt{1 - \dfrac{x}{2}}$

63. $y = (x - 1)^3 + 2$

64. $y = (1 - x)^3 + 2$

65. $y = \dfrac{1}{2x} - 1$

66. $y = \dfrac{2}{x^2} + 1$

67. $y = -\sqrt[3]{x}$

68. $y = (-2x)^{2/3}$

69. Graph the function $y = |x^2 - 1|$.

70. Graph the function $y = \sqrt{|x|}$.

Ellipses

Exercises 71–76 give equations of ellipses. Put each equation in standard form and sketch the ellipse.

71. $9x^2 + 25y^2 = 225$ **72.** $16x^2 + 7y^2 = 112$

73. $3x^2 + (y - 2)^2 = 3$ **74.** $(x + 1)^2 + 2y^2 = 4$

75. $3(x - 1)^2 + 2(y + 2)^2 = 6$

76. $6\left(x + \frac{3}{2}\right)^2 + 9\left(y - \frac{1}{2}\right)^2 = 54$

77. Write an equation for the ellipse $(x^2/16) + (y^2/9) = 1$ shifted 4 units to the left and 3 units up. Sketch the ellipse and identify its center and major axis.

78. Write an equation for the ellipse $(x^2/4) + (y^2/25) = 1$ shifted 3 units to the right and 2 units down. Sketch the ellipse and identify its center and major axis.

Even and Odd Functions

79. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $\mathbb{R}$. Which of the following (where defined) are even? odd?

a. fg **b.** f/g **c.** g/f

d. $f^2 = ff$ **e.** $g^2 = gg$ **f.** $f \circ g$

g. $g \circ f$ **h.** $f \circ f$ **i.** $g \circ g$

80. Can a function be both even and odd? Give reasons for your answer.

T **81.** (*Continuation of Example* 1.) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1 - x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

T **82.** Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

1.6 Trigonometric Functions

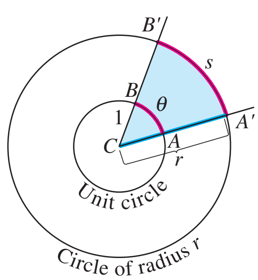


FIGURE 1.63 The radian measure of angle ACB is the length θ of arc AB on the unit circle centered at C. The value of θ can be found from any other circle, however, as the ratio s/r. Thus $s = r\theta$ is the length of arc on a circle of radius r when θ is measured in radians.

This section reviews the basic trigonometric functions. The trigonometric functions are important because they are periodic, or repeating, and therefore model many naturally occurring periodic processes.

Radian Measure

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called *radians* because of the way they simplify later calculations.

The **radian measure** of the angle ACB at the center of the unit circle (Figure 1.63) equals the length of the arc that ACB cuts from the unit circle. Figure 1.63 shows that $s = r\theta$ is the **length of arc** cut from a circle of radius r when the subtending angle θ producing the arc is measured in radians.

Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by

$$\pi \text{ radians} = 180°.$$

For example, 45° in radian measure is

$$45 \cdot \frac{\pi}{180} = \frac{\pi}{4} \text{ rad},$$

and $\pi/6$ radians is

$$\frac{\pi}{6} \cdot \frac{180}{\pi} = 30°.$$

Figure 1.64 shows the angles of two common triangles in both measures.

An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis (Figure 1.65). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180} (\approx 0.02) \text{ radians}$$

Degrees to radians: multiply by $\frac{\pi}{180}$

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57) \text{ degrees}$$

Radians to degrees: multiply by $\frac{180}{\pi}$

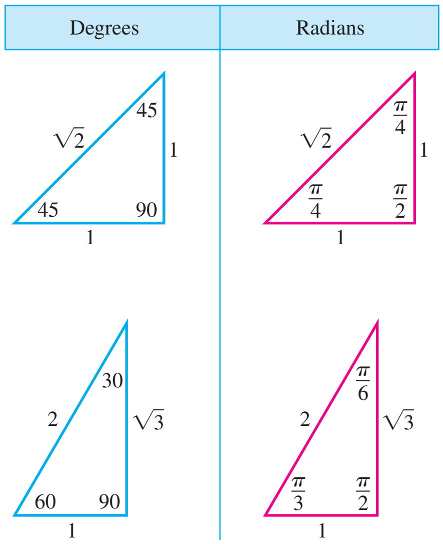


FIGURE 1.64 The angles of two common triangles, in degrees and radians.

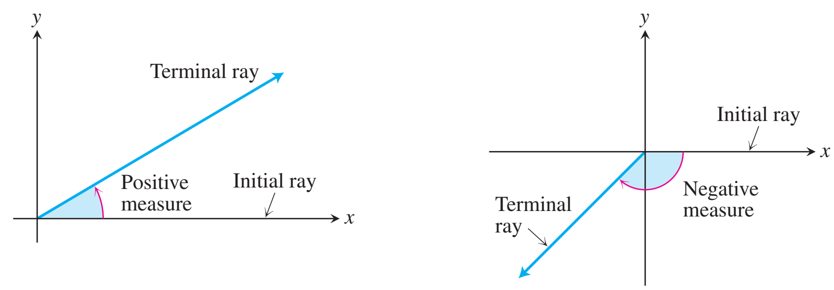


FIGURE 1.65 Angles in standard position in the *xy*-plane.

When angles are used to describe counterclockwise rotations, our measurements can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes (Figure 1.66).

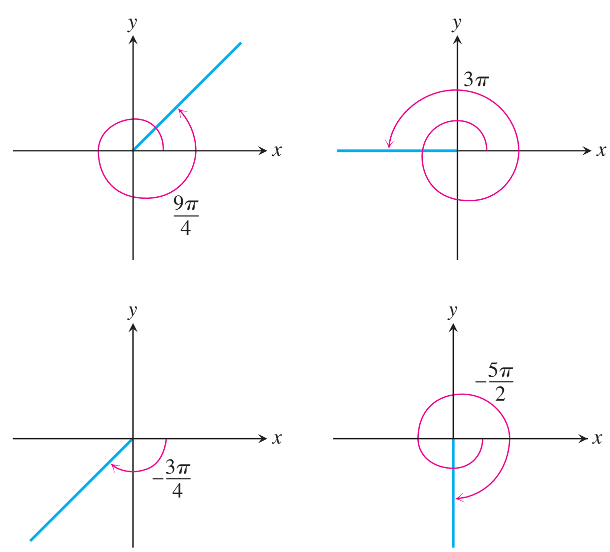
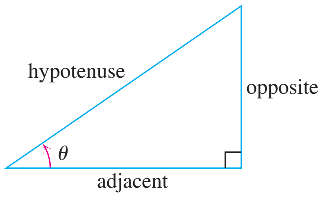


FIGURE 1.66 Nonzero radian measures can be positive or negative and can go beyond 2π.



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.67 Trigonometric ratios of an acute angle.

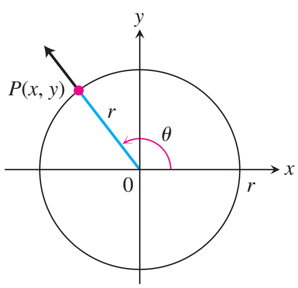


FIGURE 1.68 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

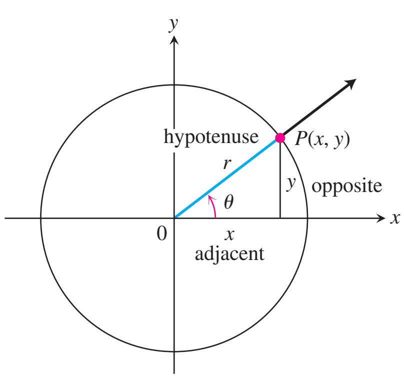


FIGURE 1.69 The new and old definitions agree for acute angles.

Angle Convention: Use Radians
From now on in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. When you do calculus, keep your calculator in radian mode.

The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 1.67). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.68).

$$\textbf{sine:}\quad \sin\theta = \frac{y}{r} \qquad \textbf{cosecant:}\quad \csc\theta = \frac{r}{y}$$
$$\textbf{cosine:}\quad \cos\theta = \frac{x}{r} \qquad \textbf{secant:}\quad \sec\theta = \frac{r}{x}$$
$$\textbf{tangent:}\quad \tan\theta = \frac{y}{x} \qquad \textbf{cotangent:}\quad \cot\theta = \frac{x}{y}$$

These extended definitions agree with the right-triangle definitions when the angle is acute (Figure 1.69).

Notice also the following definitions, whenever the quotients are defined.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = 0$. This means they are not defined if θ is $\pm\pi/2, \pm 3\pi/2, \ldots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.64. For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

The CAST rule (Figure 1.70) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure 1.71, we see that

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}.$$

Using a similar method we determined the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ shown in Table 1.4.

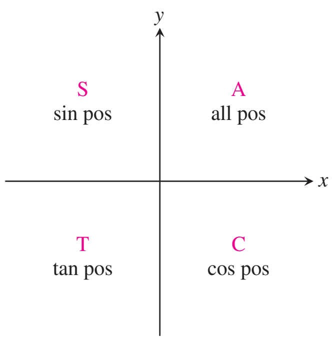


FIGURE 1.70 The CAST rule, remembered by the statement "All Students Take Calculus," tells which trigonometric functions are positive in each quadrant.

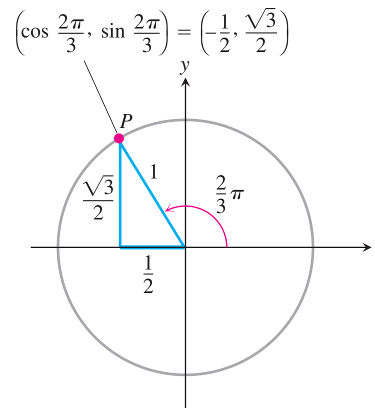


FIGURE 1.71 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

Most calculators and computers readily provide values of the trigonometric functions for angles given in either radians or degrees.

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ **(radians)**	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

EXAMPLE 1 Finding Trigonometric Function Values

If $\tan\theta = 3/2$ and $0 < \theta < \pi/2$, find the five other trigonometric functions of θ.

Solution From $\tan\theta = 3/2$, we construct the right triangle of height 3 (opposite) and base 2 (adjacent) in Figure 1.72. The Pythagorean theorem gives the length of the hypotenuse, $\sqrt{4 + 9} = \sqrt{13}$. From the triangle we write the values of the other five trigonometric functions:

$$\cos\theta = \frac{2}{\sqrt{13}}, \qquad \sin\theta = \frac{3}{\sqrt{13}}, \qquad \sec\theta = \frac{\sqrt{13}}{2}, \qquad \csc\theta = \frac{\sqrt{13}}{3}, \qquad \cot\theta = \frac{2}{3}$$

■

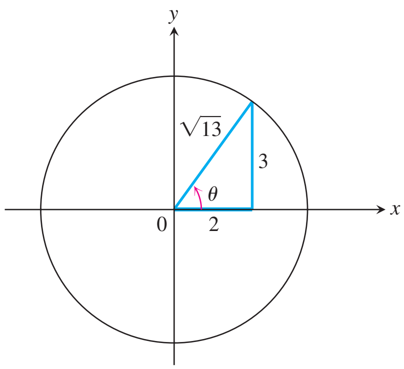


FIGURE 1.72 The triangle for calculating the trigonometric functions in Example 1.

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$$\cos(\theta + 2\pi) = \cos\theta \qquad \sin(\theta + 2\pi) = \sin\theta \qquad \tan(\theta + 2\pi) = \tan\theta$$
$$\sec(\theta + 2\pi) = \sec\theta \qquad \csc(\theta + 2\pi) = \csc\theta \qquad \cot(\theta + 2\pi) = \cot\theta$$

Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

> **DEFINITION Periodic Function**
> A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. See Figure 1.73.

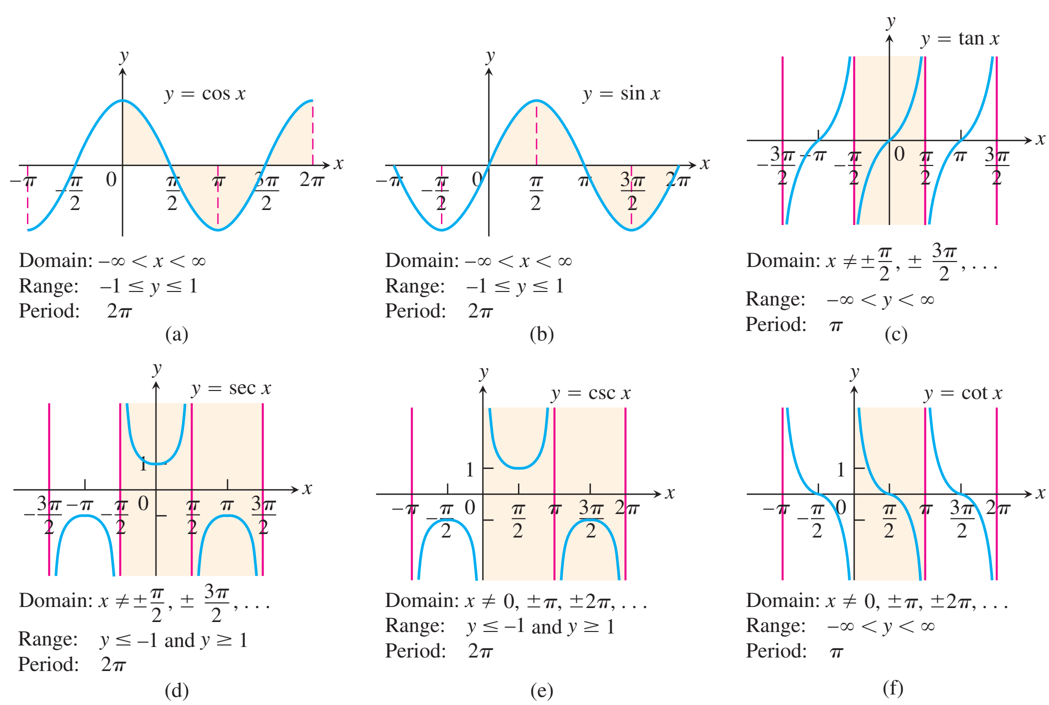


FIGURE 1.73 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

As we can see in Figure 1.73, the tangent and cotangent functions have period $p = \pi$. The other four functions have period 2π. Periodic functions are important because many behaviors studied in science are approximately periodic. A theorem from advanced calculus says that every periodic function we want to use in mathematical modeling can be written as an algebraic combination of sines and cosines. We show how to do this in Section 11.11.

The symmetries in the graphs in Figure 1.73 reveal that the cosine and secant functions are even and the other four functions are odd:

Even	**Odd**
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

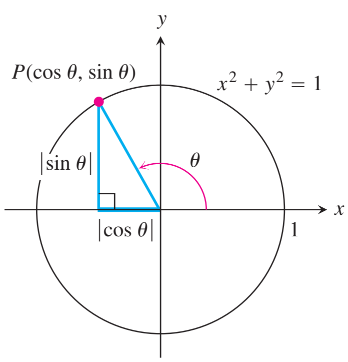


FIGURE 1.74 The reference triangle for a general angle θ.

Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (Figure 1.69). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.74 and obtain the equation

$$\cos^2\theta + \sin^2\theta = 1. \tag{1}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

The following formulas hold for all angles A and B (Exercises 53 and 54).

Addition Formulas

$$\begin{aligned}\cos(A + B) &= \cos A\cos B - \sin A\sin B\\ \sin(A + B) &= \sin A\cos B + \cos A\sin B\end{aligned} \tag{2}$$

There are similar formulas for $\cos(A - B)$ and $\sin(A - B)$ (Exercises 35 and 36). All the trigonometric identities needed in this book derive from Equations (1) and (2). For example, substituting θ for both A and B in the addition formulas gives

Double-Angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned} \tag{3}$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

We add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{4}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{5}$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{6}$$

This equation is called the **law of cosines**.

We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x-axis along one side of the triangle, as in Figure 1.75. The coordinates of A are $(b, 0)$; the coordinates of B are $(a\cos\theta, a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

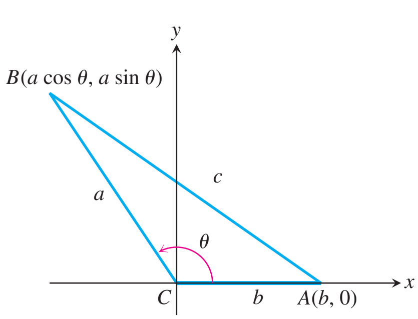


FIGURE 1.75 The square of the distance between A and B gives the law of cosines.

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

Transformations of Trigonometric Graphs

The rules for shifting, stretching, compressing, and reflecting the graph of a function apply to the trigonometric functions. The following diagram will remind you of the controlling parameters.

Vertical stretch or compression; reflection about x-axis if negative

Vertical shift

$$y = af(b(x + c)) + d$$

Horizontal stretch or compression; reflection about y-axis if negative

Horizontal shift

EXAMPLE 2 Modeling Temperature in Alaska

The builders of the Trans-Alaska Pipeline used insulated pads to keep the pipeline heat from melting the permanently frozen soil beneath. To design the pads, it was necessary to take into account the variation in air temperature throughout the year. The variation was represented in the calculations by a **general sine function** or **sinusoid** of the form

$$f(x) = A \sin\left[\frac{2\pi}{B}(x - C)\right] + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift* (Figure 1.76).

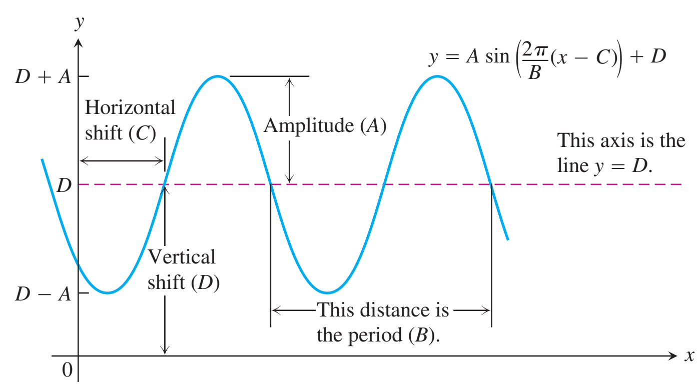


FIGURE 1.76 The general sine curve $y = A \sin[(2\pi/B)(x - C)] + D$, shown for A, B, C, and D positive (Example 2).

Figure 1.77 shows how to use such a function to represent temperature data. The data points in the figure are plots of the mean daily air temperatures for Fairbanks, Alaska, based on records of the National Weather Service from 1941 to 1970. The sine function used to fit the data is

$$f(x) = 37 \sin\left[\frac{2\pi}{365}(x - 101)\right] + 25,$$

where f is temperature in degrees Fahrenheit and x is the number of the day counting from the beginning of the year. The fit, obtained by using the sinusoidal regression feature on a calculator or computer, as we discuss in the next section, is very good at capturing the trend of the data.

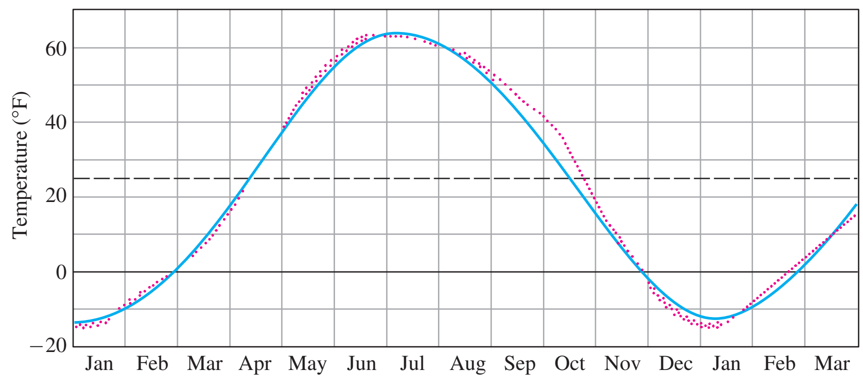


FIGURE 1.77 Normal mean air temperatures for Fairbanks, Alaska, plotted as data points (red). The approximating sine function (blue) is

$$f(x) = 37 \sin\left[(2\pi/365)(x - 101)\right] + 25.$$

EXERCISES 1.6

Radians, Degrees, and Circular Arcs

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of (a) $4\pi/5$ radians? (b) $110°$?
2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measures.
3. You want to make an $80°$ angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?
4. If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

5. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

6. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \dfrac{3}{5}, \quad x \in \left[\dfrac{\pi}{2}, \pi\right]$
8. $\tan x = 2, \quad x \in \left[0, \dfrac{\pi}{2}\right]$
9. $\cos x = \dfrac{1}{3}, \quad x \in \left[-\dfrac{\pi}{2}, 0\right]$
10. $\cos x = -\dfrac{5}{13}, \quad x \in \left[\dfrac{\pi}{2}, \pi\right]$
11. $\tan x = \dfrac{1}{2}, \quad x \in \left[\pi, \dfrac{3\pi}{2}\right]$
12. $\sin x = -\dfrac{1}{2}, \quad x \in \left[\pi, \dfrac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$
14. $\sin(x/2)$

15. $\cos \pi x$

16. $\cos \dfrac{\pi x}{2}$

17. $-\sin \dfrac{\pi x}{3}$

18. $-\cos 2\pi x$

19. $\cos\left(x - \dfrac{\pi}{2}\right)$

20. $\sin\left(x + \dfrac{\pi}{2}\right)$

21. $\sin\left(x - \dfrac{\pi}{4}\right) + 1$

22. $\cos\left(x + \dfrac{\pi}{4}\right) - 1$

Graph the functions in Exercises 23–26 in the *ts*-plane (*t*-axis horizontal, *s*-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot 2t$

24. $s = -\tan \pi t$

25. $s = \sec\left(\dfrac{\pi t}{2}\right)$

26. $s = \csc\left(\dfrac{t}{2}\right)$

T **27.** **a.** Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \le x \le 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.

b. Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \le x \le 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.

T **28.** Graph $y = \tan x$ and $y = \cot x$ together for $-7 \le x \le 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.

29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?

30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Additional Trigonometric Identities

Use the addition formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$

32. $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \dfrac{\pi}{2}\right) = \cos x$

34. $\sin\left(x - \dfrac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (Exercise 53 provides a different derivation.)

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

37. What happens if you take $B = A$ in the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?

38. What happens if you take $B = 2\pi$ in the addition formulas? Do the results agree with something you already know?

Using the Addition Formulas

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$

40. $\sin(2\pi - x)$

41. $\sin\left(\dfrac{3\pi}{2} - x\right)$

42. $\cos\left(\dfrac{3\pi}{2} + x\right)$

43. Evaluate $\sin \dfrac{7\pi}{12}$ as $\sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{3}\right)$.

44. Evaluate $\cos \dfrac{11\pi}{12}$ as $\cos\left(\dfrac{\pi}{4} + \dfrac{2\pi}{3}\right)$.

45. Evaluate $\cos \dfrac{\pi}{12}$.

46. Evaluate $\sin \dfrac{5\pi}{12}$.

Using the Double-Angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2 \dfrac{\pi}{8}$

48. $\cos^2 \dfrac{\pi}{12}$

49. $\sin^2 \dfrac{\pi}{12}$

50. $\sin^2 \dfrac{\pi}{8}$

Theory and Examples

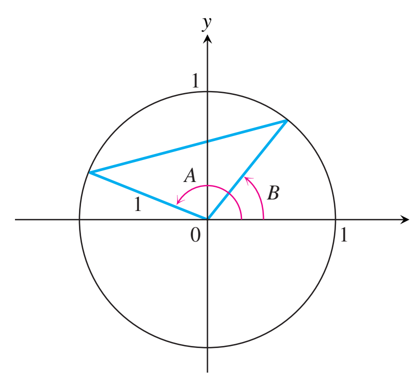
51. **The tangent sum formula** The standard formula for the tangent of the sum of two angles is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

52. *(Continuation of Exercise 51.)* Derive a formula for $\tan(A - B)$.

53. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



54. **a.** Apply the formula for $\cos(A - B)$ to the identity $\sin\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)$ to obtain the addition formula for $\sin(A + B)$.

b. Derive the formula for $\cos(A + B)$ by substituting $-B$ for B in the formula for $\cos(A - B)$ from Exercise 35.

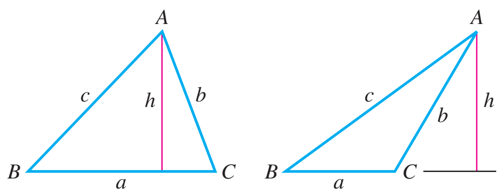
55. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side c.

56. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find the length of side c.

57. The law of sines The *law of sines* says that if a, b, and c are the sides opposite the angles A, B, and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin\theta$, if required, to derive the law.



58. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$ (as in Exercise 55). Find the sine of angle B using the law of sines.

59. A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A.

T **60. The approximation $\sin x \approx x$** It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x. In Section 3.8, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

a. With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?

b. With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

c. A quick radian mode check Is your calculator in radian mode? Evaluate $\sin x$ at a value of x near the origin, say $x = 0.1$. If $\sin x \approx x$, the calculator is in radian mode; if not, it isn't. Try it.

General Sine Curves

For

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

identify A, B, C, and D for the sine functions in Exercises 61–64 and sketch their graphs (see Figure 1.76).

61. $y = 2\sin(x + \pi) - 1$

62. $y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2}$

63. $y = -\frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}$

64. $y = \frac{L}{2\pi}\sin\frac{2\pi t}{L}, \quad L > 0$

65. Temperature in Fairbanks, Alaska Find the **(a)** amplitude, **(b)** period, **(c)** horizontal shift, and **(d)** vertical shift of the general sine function

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25.$$

66. Temperature in Fairbanks, Alaska Use the equation in Exercise 65 to approximate the answers to the following questions about the temperature in Fairbanks, Alaska, shown in Figure 1.77. Assume that the year has 365 days.

a. What are the highest and lowest mean daily temperatures shown?

b. What is the average of the highest and lowest mean daily temperatures shown? Why is this average the vertical shift of the function?

COMPUTER EXPLORATIONS

In Exercises 67–70, you will explore graphically the general sine function

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A, B, C, and D. Use a CAS or computer grapher to perform the steps in the exercises.

67. The period B Set the constants $A = 3$, $C = D = 0$.

a. Plot $f(x)$ for the values $B = 1, 3, 2\pi, 5\pi$ over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as the period increases.

b. What happens to the graph for negative values of B? Try it with $B = -3$ and $B = -2\pi$.

68. The horizontal shift C Set the constants $A = 3$, $B = 6$, $D = 0$.

a. Plot $f(x)$ for the values $C = 0, 1$, and 2 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.

b. What happens to the graph for negative values of C?

c. What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.

69. The vertical shift D Set the constants $A = 3$, $B = 6$, $C = 0$.

a. Plot $f(x)$ for the values $D = 0, 1$, and 3 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.

b. What happens to the graph for negative values of D?

70. The amplitude A Set the constants $B = 6$, $C = D = 0$.

a. Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1, 5$, and 9.

b. What happens to the graph for negative values of A?

1.7 Graphing with Calculators and Computers

A graphing calculator or a computer with graphing software enables us to graph very complicated functions with high precision. Many of these functions could not otherwise be easily graphed. However, care must be taken when using such devices for graphing purposes and we address those issues in this section. In Chapter 4 we will see how calculus helps us to be certain we are viewing accurately all the important features of a function's graph.

Graphing Windows

When using a graphing calculator or computer as a graphing tool, a portion of the graph is displayed in a rectangular **display** or **viewing window**. Often the default window gives an incomplete or misleading picture of the graph. We use the term *square window* when the units or scales on both axis are the same. This term does not mean that the display window itself is square in shape (usually it is rectangular), but means instead that the x-unit is the same as the y-unit.

When a graph is displayed in the default window, the x-unit may differ from the y-unit of scaling in order to fit the graph in the display. The viewing window in the display is set by specifying the minimum and maximum values of the independent and dependent variables. That is, an interval $a \leq x \leq b$ is specified as well as a range $c \leq y \leq d$. The machine selects a certain number of equally spaced values of x between a and b. Starting with a first value for x, if it lies within the domain of the function f being graphed, and if $f(x)$ lies inside the range $[c, d]$, then the point $(x, f(x))$ is plotted. If x lies outside the domain of f, or $f(x)$ lies outside the specified range $[c, d]$, the machine just moves on to the next x-value since it cannot plot $(x, f(x))$ in that case. The machine plots a large number of points $(x, f(x))$ in this way and approximates the curve representing the graph by drawing a short line segment between each plotted point and its next neighboring point, as we might do by hand. Usually, adjacent points are so close together that the graphical representation has the appearance of a smooth curve. Things can go wrong with this procedure and we illustrate the most common problems through the following examples.

EXAMPLE 1 Choosing a Viewing Window

Graph the function $f(x) = x^3 - 7x^2 + 28$ in each of the following display or viewing windows:

(a) $[-10, 10]$ by $[-10, 10]$ **(b)** $[-4, 4]$ by $[-50, 10]$ **(c)** $[-4, 10]$ by $[-60, 60]$

Solution

(a) We select $a = -10$, $b = 10$, $c = -10$, and $d = 10$ to specify the interval of x-values and the range of y-values for the window. The resulting graph is shown in Figure 1.78a. It appears that the window is cutting off the bottom part of the graph and that the interval of x-values is too large. Let's try the next window.

(b) Now we see more features of the graph (Figure 1.78b), but the top is missing and we need to view more to the right of $x = 4$ as well. The next window should help.

(c) Figure 1.78c shows the graph in this new viewing window. Observe that we get a more complete picture of the graph in this window and it is a reasonable graph of a third-degree polynomial. Choosing a good viewing window is a trial-and-error process which may require some troubleshooting as well.

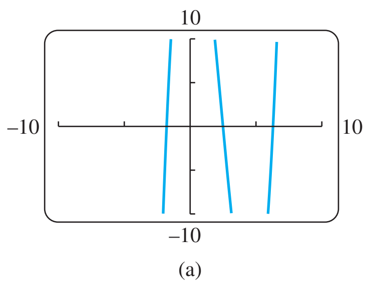


FIGURE 1.78 The graph of $f(x) = x^3 - 7x^2 + 28$ in different viewing windows (Example 1).

EXAMPLE 2 Square Windows

When a graph is displayed, the x-unit may differ from the y-unit, as in the graphs shown in Figures 1.78b and 1.78c. The result is distortion in the picture, which may be misleading. The display window can be made square by compressing or stretching the units on one axis to match the scale on the other, giving the true graph. Many systems have built-in functions to make the window "square." If yours does not, you will have to do some calculations and set the window size manually to get a square window, or bring to your viewing some foreknowledge of the true picture.

Figure 1.79a shows the graphs of the perpendicular lines $y = x$ and $y = -x + 3\sqrt{2}$, together with the semicircle $y = \sqrt{9 - x^2}$, in a nonsquare $[-6, 6]$ by $[-6, 8]$ display window. Notice the distortion. The lines do not appear to be perpendicular, and the semicircle appears to be elliptical in shape.

Figure 1.79b shows the graphs of the same functions in a square window in which the x-units are scaled to be the same as the y-units. Notice that the $[-6, 6]$ by $[-4, 4]$ viewing window has the same x-axis in both figures, but the scaling on the x-axis has been compressed in Figure 1.79b to make the window square. Figure 1.79c gives an enlarged view with a square $[-3, 3]$ by $[0, 4]$ window.

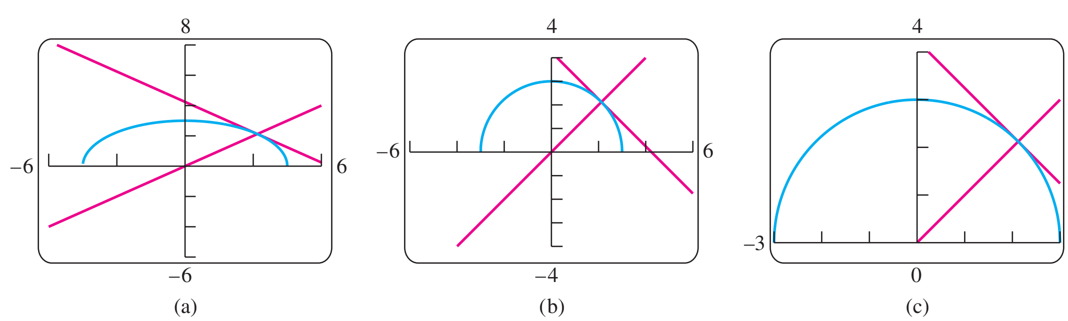


FIGURE 1.79 Graphs of the perpendicular lines $y = x$ and $y = -x + 3\sqrt{2}$, and the semicircle $y = \sqrt{9 - x^2}$, in (a) a nonsquare window, and (b) and (c) square windows (Example 2).

If the denominator of a rational function is zero at some x-value within the viewing window, a calculator or graphing computer software may produce a steep near-vertical line segment from the top to the bottom of the window. Here is an example.

EXAMPLE 3 Graph of a Rational Function

Graph the function $y = \dfrac{1}{2 - x}$.

Solution Figure 1.80a shows the graph in the $[-10, 10]$ by $[-10, 10]$ default square window with our computer graphing software. Notice the near-vertical line segment at $x = 2$. It is not truly a part of the graph and $x = 2$ does not belong to the domain of the function. By trial and error we can eliminate the line by changing the viewing window to the smaller $[-6, 6]$ by $[-4, 4]$ view, revealing a better graph (Figure 1.80b).

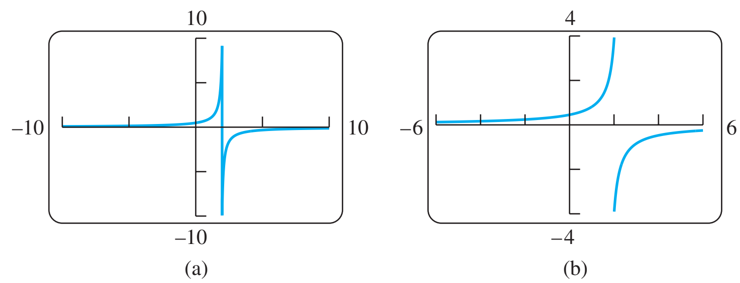


FIGURE 1.80 Graphs of the function $y = \dfrac{1}{2 - x}$ (Example 3). ■

Sometimes the graph of a trigonometric function oscillates very rapidly. When a calculator or computer software plots the points of the graph and connects them, many of the maximum and minimum points are actually missed. The resulting graph is then very misleading.

EXAMPLE 4 Graph of a Rapidly Oscillating Function

Graph the function $f(x) = \sin 100x$.

Solution Figure 1.81a shows the graph of f in the viewing window $[-12, 12]$ by $[-1, 1]$. We see that the graph looks very strange because the sine curve should oscillate periodically between -1 and 1. This behavior is not exhibited in Figure 1.81a. We might experiment with a smaller viewing window, say $[-6, 6]$ by $[-1, 1]$, but the graph is not better (Figure 1.81b). The difficulty is that the period of the trigonometric function $y = \sin 100x$ is very small $(2\pi/100 \approx 0.063)$. If we choose the much smaller viewing window $[-0.1, 0.1]$ by $[-1, 1]$ we get the graph shown in Figure 1.81c. This graph reveals the expected oscillations of a sine curve.

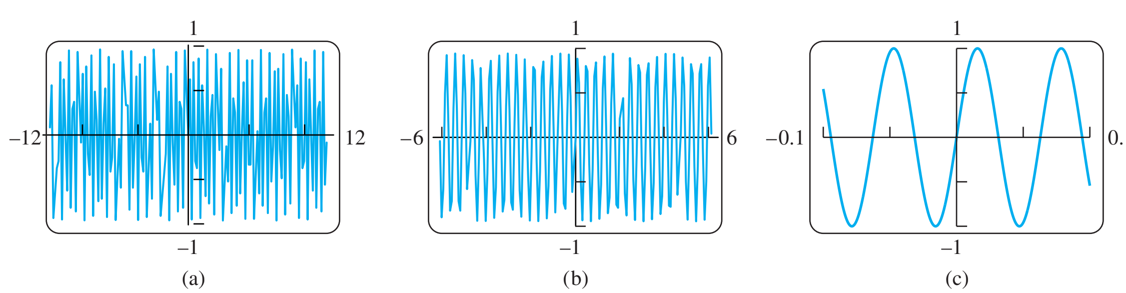


FIGURE 1.81 Graphs of the function $y = \sin 100x$ in three viewing windows. Because the period is $2\pi/100 \approx 0.063$, the smaller window in (c) best displays the true aspects of this rapidly oscillating function (Example 4). ■

EXAMPLE 5 Another Rapidly Oscillating Function

Graph the function $y = \cos x + \frac{1}{50} \sin 50x$.

Solution In the viewing window $[-6, 6]$ by $[-1, 1]$ the graph appears much like the cosine function with some small sharp wiggles on it (Figure 1.82a). We get a better look when we significantly reduce the window to $[-0.6, 0.6]$ by $[0.8, 1.02]$, obtaining the graph in Figure 1.82b. We now see the small but rapid oscillations of the second term, $1/50 \sin 50x$, added to the comparatively larger values of the cosine curve.

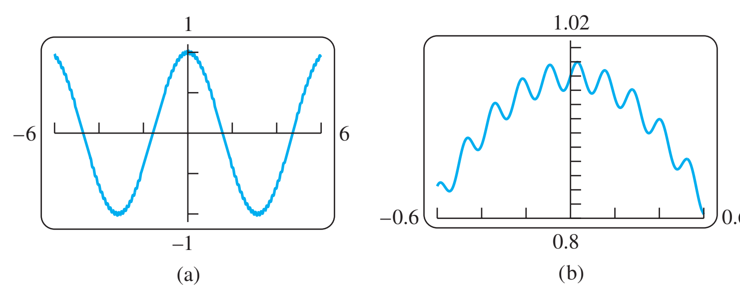


FIGURE 1.82 In (b) we see a close-up view of the function $y = \cos x + \frac{1}{50} \sin 50x$ graphed in (a). The term $\cos x$ clearly dominates the second term, $\frac{1}{50} \sin 50x$, which produces the rapid oscillations along the cosine curve (Example 5). ■

EXAMPLE 6 Graphing an Odd Fractional Power

Graph the function $y = x^{1/3}$.

Solution Many graphing devices display the graph shown in Figure 1.83a. When we compare it with the graph of $y = x^{1/3} = \sqrt[3]{x}$ in Figure 1.38, we see that the left branch for $x < 0$ is missing. The reason the graphs differ is that many calculators and computer soft-

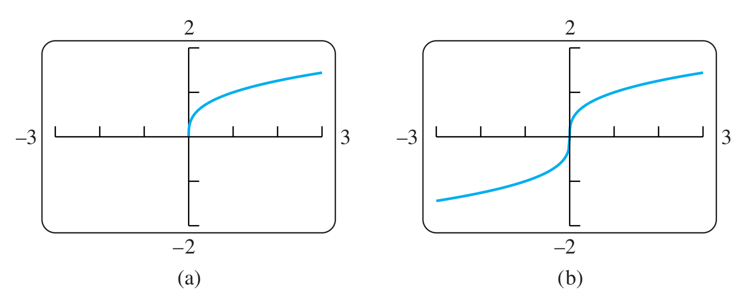


FIGURE 1.83 The graph of $y = x^{1/3}$ is missing the left branch in (a). In (b) we graph the function $f(x) = \frac{x}{|x|} \cdot |x|^{1/3}$ obtaining both branches. (See Example 6.)

ware programs calculate $x^{1/3}$ as $e^{(1/3)\ln x}$. (The exponential and logarithmic functions are studied in Chapter 7.) Since the logarithmic function is not defined for negative values of x, the computing device can only produce the right branch where $x > 0$.

To obtain the full picture showing both branches, we can graph the function

$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}.$$

This function equals $x^{1/3}$ except at $x = 0$ (where f is undefined, although $0^{1/3} = 0$). The graph of f is shown in Figure 1.83b. ■

Empirical Modeling: Capturing the Trend of Collected Data

In Example 3 of Section 1.4, we verified the reasonableness of Kepler's hypothesis that the period of a planet's orbit is proportional to its mean distance from the sun raised to the 3/2 power. If we cannot hypothesize a relationship between a dependent variable and an independent variable, we might collect data points and try to find a curve that "fits" the data and captures the trend of the scatterplot. The process of finding a curve to fit data is called **regression analysis** and the curve is called a **regression curve**. A computer or graphing calculator finds the regression curve by finding the particular curve which minimizes the sum of the squares of the vertical distances between the data points and the curve. This method of **least squares** is discussed in the Section 14.7 exercises.

There are many useful types of regression curves, such as straight lines, power, polynomial, exponential, logarithmic, and sinusoidal curves. Many computers or graphing calculators have a regression analysis feature to fit a variety of regression curve types. The next example illustrates using a graphing calculator's linear regression feature to fit data from Table 1.5 with a linear equation.

EXAMPLE 7 Fitting a Regression Line

Starting with the data in Table 1.5, build a model for the price of a postage stamp as a function of time. After verifying that the model is "reasonable," use it to predict the price in 2010.

TABLE 1.5 Price of a U.S. postage stamp

Year x	Cost y
1968	0.06
1971	0.08
1974	0.10
1975	0.13
1977	0.15
1981	0.18
1981	0.20
1985	0.22
1987	0.25
1991	0.29
1995	0.32
1998	0.33
2002	0.37

Solution We are building a model for the price of a stamp since 1968. There were two increases in 1981, one of three cents followed by another of two cents. To make 1981 comparable with the other listed years, we lump them together as a single five-cent increase, giving the data in Table 1.6. Figure 1.84a gives the scatterplot for Table 1.6.

TABLE 1.6 Price of a U.S postage stamp since 1968

x	0	3	6	7	9	13	17	19	23	27	30	34
y	6	8	10	13	15	20	22	25	29	32	33	37

Since the scatterplot is fairly linear, we investigate a linear model. Upon entering the data into a graphing calculator (or computer software) and selecting the linear regression option, we find the regression line to be

$$y = 0.94x + 6.10.$$

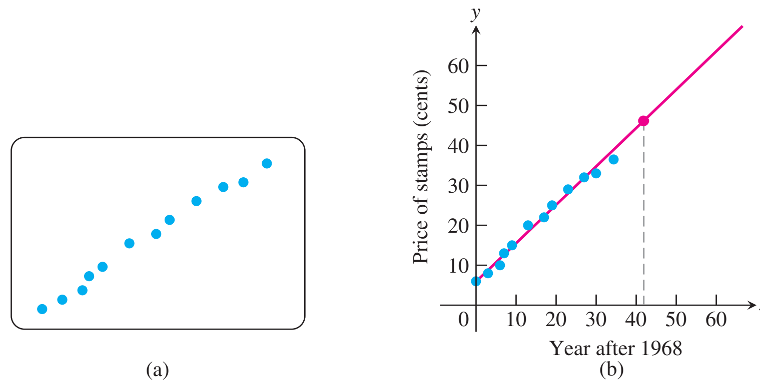


FIGURE 1.84 (a) Scatterplot of (x, y) data in Table 1.6. (b) Using the regression line to estimate the price of a stamp in 2010. (Example 7).

Figure 1.84b shows the line and scatterplot together. The fit is remarkably good, so the model seems reasonable.

Evaluating the regression line, we conclude that in 2010 $(x = 42)$, the price of a stamp will be

$$y = 0.94(42) + 6.10 \approx 46 \text{ cents}.$$

The prediction is shown as the red point on the regression line in Figure 1.84b. ■

EXAMPLE 8 Finding a Curve to Predict Population Levels

We may want to predict the future size of a population, such as the number of trout or catfish living in a fish farm. Figure 1.85 shows a scatterplot of the data collected by R. Pearl for a collection of yeast cells (measured as **biomass**) growing over time (measured in hours) in a nutrient.

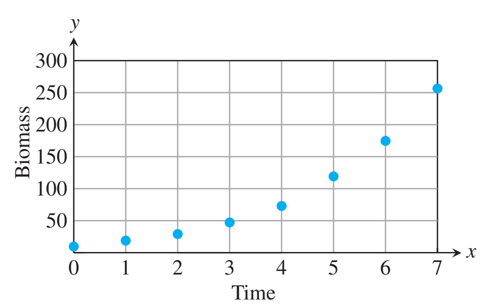


FIGURE 1.85 Biomass of a yeast culture versus elapsed time (Example 8).
(Data from R. Pearl, "The Growth of Population," *Quart. Rev. Biol.*, Vol. 2 (1927), pp. 532–548.)

The plot of points appears to be reasonably smooth with an upward curving trend. We might attempt to capture this trend by fitting a polynomial (for example, a quadratic $y = ax^2 + bx + c$), a power curve $(y = ax^b)$, or an exponential curve $(y = ae^{bx})$. Figure 1.86 shows the result of using a calculator to fit a quadratic model.

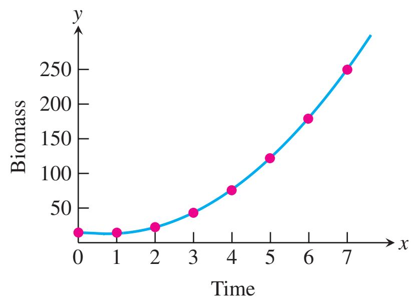


FIGURE 1.86 Fitting a quadratic to Pearl's data gives the equation $y = 6.10x^2 - 9.28x + 16.43$ and the prediction $y(17) = 1622.65$ (Example 8).

The quadratic model $y = 6.10x^2 - 9.28x + 16.43$ appears to fit the collected data reasonably well (Figure 1.86). Using this model, we predict the population after 17 hours as $y(17) = 1622.65$. Let us examine more of Pearl's data to see if our quadratic model continues to be a good one.

In Figure 1.87, we display all of Pearl's data. Now you see that the prediction of $y(17) = 1622.65$ grossly overestimates the observed population of 659.6. Why did the quadratic model fail to predict a more accurate value?

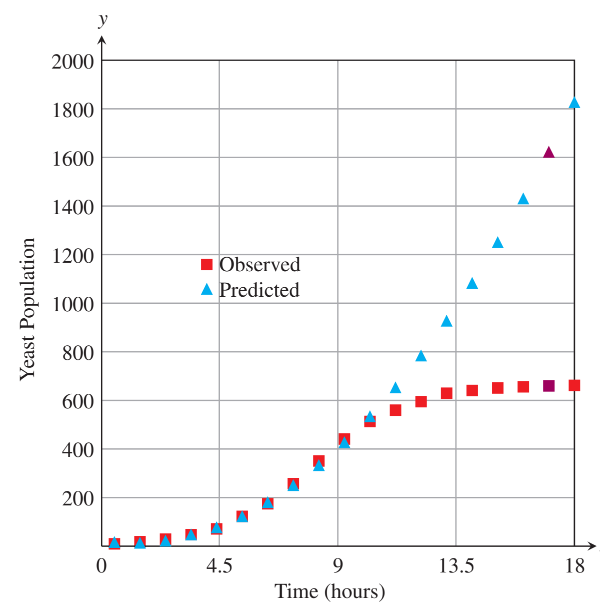


FIGURE 1.87 The rest of Pearl's data (Example 8).

The problem lies in the danger of predicting beyond the range of data used to build the empirical model. (The range of data creating our model was $0 \leq x \leq 7$.) Such *extrapolation* is especially dangerous when the model selected is not supported by some underlying rationale suggesting the form of the model. In our yeast example, why would we expect a quadratic function as underlying population growth? Why not an exponential function? In the face of this, how then do we predict future values? Often, calculus can help, and in Chapter 9 we use it to model population growth. ■

Regression Analysis

Regression analysis has four steps:

1. Plot the data (scatterplot).
2. Find a regression equation. For a line, it has the form $y = mx + b$, and for a quadratic, the form $y = ax^2 + bx + c$.
3. Superimpose the graph of the regression equation on the scatterplot to see the fit.
4. If the fit is satisfactory, use the regression equation to predict y-values for values of x not in the table.

EXERCISES 1.7

Choosing a Viewing Window

In Exercises 1–4, use a graphing calculator or computer to determine which of the given viewing windows displays the most appropriate graph of the specified function.

1. $f(x) = x^4 - 7x^2 + 6x$
   a. $[-1, 1]$ by $[-1, 1]$ b. $[-2, 2]$ by $[-5, 5]$
   c. $[-10, 10]$ by $[-10, 10]$ d. $[-5, 5]$ by $[-25, 15]$
2. $f(x) = x^3 - 4x^2 - 4x + 16$
   a. $[-1, 1]$ by $[-5, 5]$ b. $[-3, 3]$ by $[-10, 10]$
   c. $[-5, 5]$ by $[-10, 20]$ d. $[-20, 20]$ by $[-100, 100]$
3. $f(x) = 5 + 12x - x^3$
   a. $[-1, 1]$ by $[-1, 1]$ b. $[-5, 5]$ by $[-10, 10]$
   c. $[-4, 4]$ by $[-20, 20]$ d. $[-4, 5]$ by $[-15, 25]$
4. $f(x) = \sqrt{5 + 4x - x^2}$
   a. $[-2, 2]$ by $[-2, 2]$ b. $[-2, 6]$ by $[-1, 4]$
   c. $[-3, 7]$ by $[0, 10]$ d. $[-10, 10]$ by $[-10, 10]$

Determining a Viewing Window

In Exercises 5–30, determine an appropriate viewing window for the given function and use it to display its graph.

5. $f(x) = x^4 - 4x^3 + 15$
6. $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + 1$
7. $f(x) = x^5 - 5x^4 + 10$
8. $f(x) = 4x^3 - x^4$
9. $f(x) = x\sqrt{9 - x^2}$
10. $f(x) = x^2(6 - x^3)$
11. $y = 2x - 3x^{2/3}$
12. $y = x^{1/3}(x^2 - 8)$
13. $y = 5x^{2/5} - 2x$
14. $y = x^{2/3}(5 - x)$
15. $y = |x^2 - 1|$
16. $y = |x^2 - x|$
17. $y = \dfrac{x + 3}{x + 2}$
18. $y = 1 - \dfrac{1}{x + 3}$
19. $f(x) = \dfrac{x^2 + 2}{x^2 + 1}$
20. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$
21. $f(x) = \dfrac{x - 1}{x^2 - x - 6}$
22. $f(x) = \dfrac{8}{x^2 - 9}$
23. $f(x) = \dfrac{6x^2 - 15x + 6}{4x^2 - 10x}$
24. $f(x) = \dfrac{x^2 - 3}{x - 2}$
25. $y = \sin 250x$
26. $y = 3\cos 60x$
27. $y = \cos\left(\dfrac{x}{50}\right)$
28. $y = \dfrac{1}{10}\sin\left(\dfrac{x}{10}\right)$
29. $y = x + \dfrac{1}{10}\sin 30x$
30. $y = x^2 + \dfrac{1}{50}\cos 100x$
31. Graph the lower half of the circle defined by the equation $x^2 + 2x = 4 + 4y - y^2$.
32. Graph the upper branch of the hyperbola $y^2 - 16x^2 = 1$.
33. Graph four periods of the function $f(x) = -\tan 2x$.
34. Graph two periods of the function $f(x) = 3\cot\dfrac{x}{2} + 1$.
35. Graph the function $f(x) = \sin 2x + \cos 3x$.
36. Graph the function $f(x) = \sin^3 x$.

Graphing in Dot Mode

Another way to avoid incorrect connections when using a graphing device is through the use of a "dot mode," which plots only the points. If your graphing utility allows that mode, use it to plot the functions in Exercises 37–40.

37. $y = \dfrac{1}{x - 3}$
38. $y = \sin\dfrac{1}{x}$
39. $y = x\lfloor x \rfloor$
40. $y = \dfrac{x^3 - 1}{x^2 - 1}$

Regression Analysis

T 41. Table 1.7 shows the mean annual compensation of construction workers.

TABLE 1.7 Construction workers' average annual compensation

Year	Annual compensation (dollars)
1980	22,033
1985	27,581
1988	30,466
1990	32,836
1992	34,815
1995	37,996
1999	42,236
2002	45,413

Source: U.S. Bureau of Economic Analysis.

a. Find a linear regression equation for the data.
b. Find the slope of the regression line. What does the slope represent?

c. Superimpose the graph of the linear regression equation on a scatterplot of the data.

d. Use the regression equation to predict the construction workers' average annual compensation in 2010.

T **42.** The median price of existing single-family homes has increased consistently since 1970. The data in Table 1.8, however, show that there have been differences in various parts of the country.

a. Find a linear regression equation for home cost in the Northeast.

b. What does the slope of the regression line represent?

c. Find a linear regression equation for home cost in the Midwest.

d. Where is the median price increasing more rapidly, in the Northeast or the Midwest?

TABLE 1.8 Median price of single-family homes

Year	Northeast (dollars)	Midwest (dollars)
1970	25,200	20,100
1975	39,300	30,100
1980	60,800	51,900
1985	88,900	58,900
1990	141,200	74,000
1995	197,100	88,300
2000	264,700	97,000

Source: National Association of Realtors®

T **43. Vehicular stopping distance** Table 1.9 shows the total stopping distance of a car as a function of its speed.

a. Find the quadratic regression equation for the data in Table 1.9.

b. Superimpose the graph of the quadratic regression equation on a scatterplot of the data.

c. Use the graph of the quadratic regression equation to predict the average total stopping distance for speeds of 72 and 85 mph. Confirm algebraically.

d. Now use *linear* regression to predict the average total stopping distance for speeds of 72 and 85 mph. Superimpose the regression line on a scatterplot of the data. Which gives the better fit, the line here or the graph in part (b)?

TABLE 1.9 Vehicular stopping distance

Speed (mph)	Average total stopping distance (ft)
20	42
25	56
30	73.5
35	91.5
40	116
45	142.5
50	173
55	209.5
60	248
65	292.5
70	343
75	401
80	464

Source: U.S. Bureau of Public Roads.

T **44. Stern waves** Observations of the stern waves that follow a boat at right angles to its course have disclosed that the distance between the crests of these waves (their *wave length*) increases with the speed of the boat. Table 1.10 shows the relationship between wave length and the speed of the boat.

TABLE 1.10 Wave lengths

Wave length (m)	Speed (km/h)
0.20	1.8
0.65	3.6
1.13	5.4
2.55	7.2
4.00	9.0
5.75	10.8
7.80	12.6
10.20	14.4
12.90	16.2
16.00	18.0
18.40	19.8

a. Find a power regression equation $y = ax^b$ for the data in Table 1.10, where x is the wave length, and y the speed of the boat.

b. Superimpose the graph of the power regression equation on a scatterplot of the data.

c. Use the graph of the power regression equation to predict the speed of the boat when the wave length is 11 m. Confirm algebraically.

d. Now use *linear* regression to predict the speed when the wave length is 11 m. Superimpose the regression line on a scatterplot of the data. Which gives the better fit, the line here or the curve in part (b)?

Chapter 1 Questions to Guide Your Review

1. How are the real numbers represented? What are the main categories characterizing the properties of the real number system? What are the primary subsets of the real numbers?

2. How are the rational numbers described in terms of decimal expansions? What are the irrational numbers? Give examples.

3. What are the order properties of the real numbers? How are they used in solving equations?

4. What is a number's absolute value? Give examples? How are $|-a|$, $|ab|$, $|a/b|$, and $|a + b|$ related to $|a|$ and $|b|$?

5. How are absolute values used to describe intervals or unions of intervals? Give examples.

6. How do we identify points in the plane using the Cartesian coordinate system? What is the graph of an equation in the variables x and y?

7. How can you write an equation for a line if you know the coordinates of two points on the line? The line's slope and the coordinates of one point on the line? The line's slope and y-intercept? Give examples.

8. What are the standard equations for lines perpendicular to the coordinate axes?

9. How are the slopes of mutually perpendicular lines related? What about parallel lines? Give examples.

10. When a line is not vertical, what is the relation between its slope and its angle of inclination?

11. How do you find the distance between two points in the coordinate plane?

12. What is the standard equation of a circle with center (h, k) and radius a? What is the unit circle and what is its equation?

13. Describe the steps you would take to graph the circle $x^2 + y^2 + 4x - 6y + 12 = 0$.

14. What inequality describes the points in the coordinate plane that lie inside the circle of radius a centered at the point (h, k)? That lie inside or on the circle? That lie outside the circle? That lie outside or on the circle?

15. If a, b, and c are constants and $a \neq 0$, what can you say about the graph of the equation $y = ax^2 + bx + c$? In particular, how would you go about sketching the curve $y = 2x^2 + 4x$?

16. What is a function? What is its domain? Its range? What is an arrow diagram for a function? Give examples.

17. What is the graph of a real-valued function of a real variable? What is the vertical line test?

18. What is a piecewise-defined function? Give examples.

19. What are the important types of functions frequently encountered in calculus? Give an example of each type.

20. In terms of its graph, what is meant by an increasing function? A decreasing function? Give an example of each.

21. What is an even function? An odd function? What symmetry properties do the graphs of such functions have? What advantage can we take of this? Given an example of a function that is neither even nor odd.

22. What does it mean to say that y is proportional to x? To $x^{3/2}$? What is the geometric interpretation of proportionality? How can this interpretation be used to test a proposed proportionality?

23. If f and g are real-valued functions, how are the domains of $f + g$, $f - g$, fg, and f/g related to the domains of f and g? Give examples.

24. When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?

25. How do you change the equation $y = f(x)$ to shift its graph vertically up or down by a factor $k > 0$? Horizontally to the left or right? Give examples.

26. How do you change the equation $y = f(x)$ to compress or stretch the graph by $c > 1$? Reflect the graph across a coordinate axis? Give examples.

27. What is the standard equation of an ellipse with center (h, k)? What is its major axis? Its minor axis? Give examples.

28. What is radian measure? How do you convert from radians to degrees? Degrees to radians?

29. Graph the six basic trigonometric functions. What symmetries do the graphs have?

30. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?

31. Starting with the identity $\sin^2\theta + \cos^2\theta = 1$ and the formulas for $\cos(A + B)$ and $\sin(A + B)$, show how a variety of other trigonometric identities may be derived.

32. How does the formula for the general sine function $f(x) = A\sin((2\pi/B)(x - C)) + D$ relate to the shifting, stretching, compressing, and reflection of its graph? Give examples. Graph the general sine curve and identify the constants A, B, C, and D.

33. Name three issues that arise when functions are graphed using a calculator or computer with graphing software. Give examples.

Chapter 1 Practice Exercises

Inequalities

In Exercises 1–4, solve the inequalities and show the solution sets on the real line.

1. $7 + 2x \geq 3$

2. $-3x < 10$

3. $\frac{1}{5}(x - 1) < \frac{1}{4}(x - 2)$

4. $\frac{x - 3}{2} \geq -\frac{4 + x}{3}$

Absolute Value

Solve the equations or inequalities in Exercises 5–8.

5. $|x + 1| = 7$

6. $|y - 3| < 4$

7. $\left|1 - \frac{x}{2}\right| > \frac{3}{2}$

8. $\left|\frac{2x + 7}{3}\right| \leq 5$

Coordinates

9. A particle in the plane moved from $A(-2, 5)$ to the y-axis in such a way that Δy equaled $3\Delta x$. What were the particle's new coordinates?

10. **a.** Plot the points $A(8, 1)$, $B(2, 10)$, $C(-4, 6)$, $D(2, -3)$, and $E(14/3, 6)$.

b. Find the slopes of the lines AB, BC, CD, DA, CE, and BD.

c. Do any four of the five points A, B, C, D, and E form a parallelogram?

d. Are any three of the five points collinear? How do you know?

e. Which of the lines determined by the five points pass through the origin?

11. Do the points $A(6, 4)$, $B(4, -3)$, and $C(-2, 3)$ form an isosceles triangle? A right triangle? How do you know?

12. Find the coordinates of the point on the line $y = 3x + 1$ that is equidistant from $(0, 0)$ and $(-3, 4)$.

Lines

In Exercises 13–24, write an equation for the specified line.

13. through $(1, -6)$ with slope 3

14. through $(-1, 2)$ with slope $-1/2$

15. the vertical line through $(0, -3)$

16. through $(-3, 6)$ and $(1, -2)$

17. the horizontal line through $(0, 2)$

18. through $(3, 3)$ and $(-2, 5)$

19. with slope -3 and y-intercept 3

20. through $(3, 1)$ and parallel to $2x - y = -2$

21. through $(4, -12)$ and parallel to $4x + 3y = 12$

22. through $(-2, -3)$ and perpendicular to $3x - 5y = 1$

23. through $(-1, 2)$ and perpendicular to $(1/2)x + (1/3)y = 1$

24. with x-intercept 3 and y-intercept -5

Functions and Graphs

25. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.

26. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.

27. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.

28. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of liftoff. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

In Exercises 29–32, determine whether the graph of the function is symmetric about the y-axis, the origin, or neither.

29. $y = x^{1/5}$

30. $y = x^{2/5}$

31. $y = x^2 - 2x - 1$

32. $y = e^{-x^2}$

In Exercises 33–40, determine whether the function is even, odd, or neither.

33. $y = x^2 + 1$

34. $y = x^5 - x^3 - x$

35. $y = 1 - \cos x$

36. $y = \sec x \tan x$

37. $y = \frac{x^4 + 1}{x^3 - 2x}$

38. $y = 1 - \sin x$

39. $y = x + \cos x$

40. $y = \sqrt{x^4 - 1}$

In Exercises 41–50, find the **(a)** domain and **(b)** range.

41. $y = |x| - 2$

42. $y = -2 + \sqrt{1 - x}$

43. $y = \sqrt{16 - x^2}$

44. $y = 3^{2-x} + 1$

45. $y = 2e^{-x} - 3$

46. $y = \tan(2x - \pi)$

47. $y = 2\sin(3x + \pi) - 1$

48. $y = x^{2/5}$

49. $y = \ln(x - 3) + 1$

50. $y = -1 + \sqrt[3]{2 - x}$

Piecewise-Defined Functions

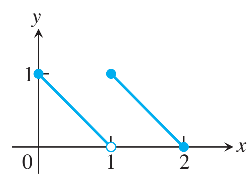
In Exercises 51 and 52, find the **(a)** domain and **(b)** range.

51. $y = \begin{cases} \sqrt{-x}, & -4 \le x \le 0 \\ \sqrt{x}, & 0 < x \le 4 \end{cases}$

52. $y = \begin{cases} -x - 2, & -2 \le x \le -1 \\ x, & -1 < x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$

In Exercises 53 and 54, write a piecewise formula for the function.

53.



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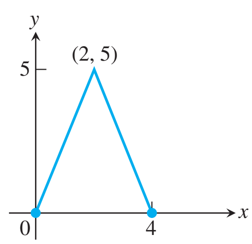


Composition of Functions

In Exercises 55 and 56, find

a. $(f \circ g)(-1)$.

b. $(g \circ f)(2)$.

c. $(f \circ f)(x)$.

d. $(g \circ g)(x)$.

55. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{\sqrt{x + 2}}$

56. $f(x) = 2 - x, \quad g(x) = \sqrt[3]{x + 1}$

In Exercises 57 and 58, **(a)** write a formula for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

57. $f(x) = 2 - x^2, \quad g(x) = \sqrt{x + 2}$

58. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{1 - x}$

Composition with absolute values In Exercises 59–64, graph f_1 and f_2 together. Then describe how applying the absolute value function before applying f_1 affects the graph.

	$f_1(x)$	$f_2(x) = f_1(\lvert x \rvert)$
59.	x	$\lvert x \rvert$
60.	x^3	$\lvert x \rvert^3$
61.	x^2	$\lvert x \rvert^2$
62.	$\frac{1}{x}$	$\frac{1}{\lvert x \rvert}$
63.	$\sqrt{x}$	$\sqrt{\lvert x \rvert}$
64.	$\sin x$	$\sin \lvert x \rvert$

Composition with absolute values In Exercises 65–68, graph g_1 and g_2 together. Then describe how taking absolute values after applying g_1 affects the graph.

	$g_1(x)$	$g_2(x) = \lvert g_1(x) \rvert$
65.	x^3	$\lvert x^3 \rvert$
66.	$\sqrt{x}$	$\lvert \sqrt{x} \rvert$
67.	$4 - x^2$	$\lvert 4 - x^2 \rvert$
68.	$x^2 + x$	$\lvert x^2 + x \rvert$

Trigonometry

In Exercises 69–72, sketch the graph of the given function. What is the period of the function?

69. $y = \cos 2x$

70. $y = \sin \dfrac{x}{2}$

71. $y = \sin \pi x$

72. $y = \cos \dfrac{\pi x}{2}$

73. Sketch the graph $y = 2\cos\left(x - \dfrac{\pi}{3}\right)$.

74. Sketch the graph $y = 1 + \sin\left(x + \dfrac{\pi}{4}\right)$.

In Exercises 75–78, ABC is a right triangle with the right angle at C. The sides opposite angles A, B, and C are a, b, and c, respectively.

75. **a.** Find a and b if $c = 2$, $B = \pi/3$.

b. Find a and c if $b = 2$, $B = \pi/3$.

76. **a.** Express a in terms of A and c.

b. Express a in terms of A and b.

77. **a.** Express a in terms of B and b.

b. Express c in terms of A and a.

78. **a.** Express $\sin A$ in terms of a and c.

b. Express $\sin A$ in terms of b and c.

79. **Height of a pole** Two wires stretch from the top T of a vertical pole to points B and C on the ground, where C is 10 m closer to the base of the pole than is B. If wire BT makes an angle of 35° with the horizontal and wire CT makes an angle of 50° with the horizontal, how high is the pole?

80. **Height of a weather balloon** Observers at positions A and B 2 km apart simultaneously measure the angle of elevation of a weather balloon to be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between A and B, find the height of the balloon.

T **81.** **a.** Graph the function $f(x) = \sin x + \cos(x/2)$.

b. What appears to be the period of this function?

c. Confirm your finding in part (b) algebraically.

T **82.** **a.** Graph $f(x) = \sin(1/x)$.

b. What are the domain and range of f?

c. Is f periodic? Give reasons for your answer.

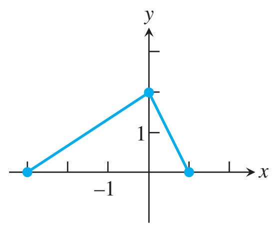
Chapter 1 Additional and Advanced Exercises

Functions and Graphs

1. The graph of f is shown. Draw the graph of each function.

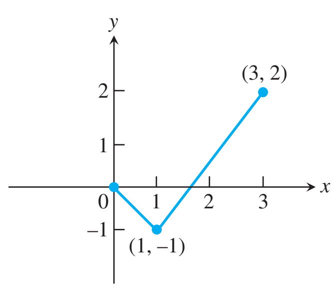
   a. $y = f(-x)$ **b.** $y = -f(x)$

   c. $y = -2f(x+1) + 1$ **d.** $y = 3f(x-2) - 2$



2. A portion of the graph of a function defined on $[-3, 3]$ is shown. Complete the graph assuming that the function is

   a. even. **b.** odd.



3. Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.
4. Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.
5. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.
6. If $g(x)$ is an odd function defined for all values of x, can anything be said about $g(0)$? Give reasons for your answer.
7. Graph the equation $|x| + |y| = 1 + x$.
8. Graph the equation $y + |y| = x + |x|$.

Trigonometry

In Exercises 9–14, ABC is an arbitrary triangle with sides a, b, and c opposite angles A, B, and C, respectively.

9. Find b if $a = \sqrt{3}$, $A = \pi/3$, $B = \pi/4$.
10. Find $\sin B$ if $a = 4$, $b = 3$, $A = \pi/4$.
11. Find $\cos A$ if $a = 2$, $b = 2$, $c = 3$.
12. Find c if $a = 2$, $b = 3$, $C = \pi/4$.
13. Find $\sin B$ if $a = 2$, $b = 3$, $c = 4$.
14. Find $\sin C$ if $a = 2$, $b = 4$, $c = 5$.

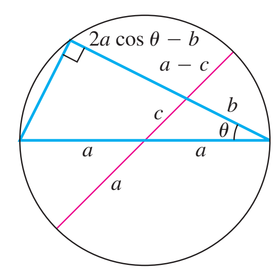
Derivations and Proofs

15. Prove the following identities.

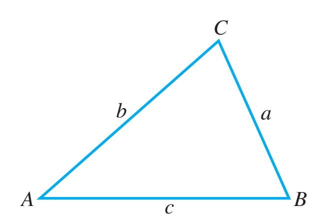
    a. $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$

    b. $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$

16. Explain the following "proof without words" of the law of cosines. (Source: "Proof without Words: The Law of Cosines," Sidney H. Kung, *Mathematics Magazine*, Vol. 63, No. 5, Dec. 1990, p. 342.)



17. Show that the area of triangle ABC is given by $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$.



18. Show that the area of triangle ABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a + b + c)/2$ is the semiperimeter of the triangle.
19. **Properties of inequalities** If a and b are real numbers, we say that ***a* is less than *b*** and write $a < b$ if (and only if) $b - a$ is positive. Use this definition to prove the following properties of inequalities.

    If a, b, and c are real numbers, then:

    1. $a < b \Rightarrow a + c < b + c$
    2. $a < b \Rightarrow a - c < b - c$
    3. $a < b$ and $c > 0 \Rightarrow ac < bc$
    4. $a < b$ and $c < 0 \Rightarrow bc < ac$
       (Special case: $a < b \Rightarrow -b < -a$)

5. $a > 0 \Rightarrow \frac{1}{a} > 0$

6. $0 < a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

7. $a < b < 0 \Rightarrow \frac{1}{b} < \frac{1}{a}$

20. Prove that the following inequalities hold for any real numbers a and b.

a. $|a| < |b|$ if and only if $a^2 < b^2$

b. $|a - b| \geq ||a| - |b||$

Generalizing the triangle inequality Prove by mathematical induction that the inequalities in Exercises 21 and 22 hold for any n real numbers $a_1, a_2, \ldots, a_n$. (Mathematical induction is reviewed in Appendix 1.)

21. $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$

22. $|a_1 + a_2 + \cdots + a_n| \geq |a_1| - |a_2| - \cdots - |a_n|$

23. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

24. a. Even-odd decompositions Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

b. Uniqueness Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 23 to show that $E = E_1$ and $O = O_1$.)

Grapher Explorations—Effects of Parameters

25. What happens to the graph of $y = ax^2 + bx + c$ as

a. a changes while b and c remain fixed?

b. b changes (a and c fixed, $a \neq 0$)?

c. c changes (a and b fixed, $a \neq 0$)?

26. What happens to the graph of $y = a(x + b)^3 + c$ as

a. a changes while b and c remain fixed?

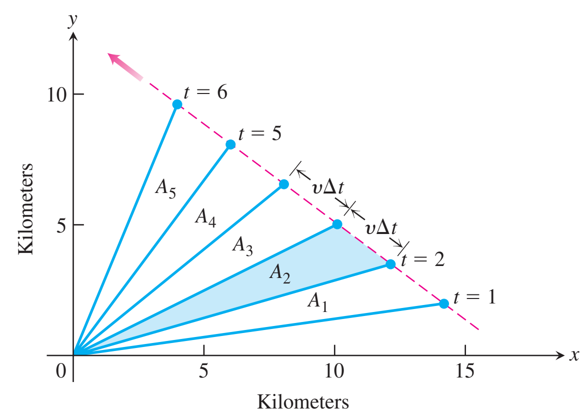
b. b changes (a and c fixed, $a \neq 0$)?

c. c changes (a and b fixed, $a \neq 0$)?

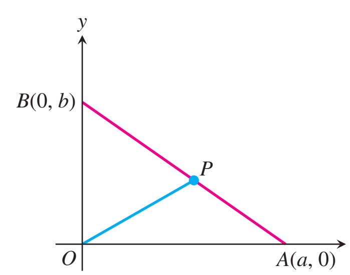
27. Find all values of the slope of the line $y = mx + 2$ for which the x-intercept exceeds $1/2$.

Geometry

28. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 13.6), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



29. a. Find the slope of the line from the origin to the midpoint P, of side AB in the triangle in the accompanying figure $(a, b > 0)$.



b. When is OP perpendicular to AB?

Chapter 1 Technology Application Projects

An Overview of Mathematica

An overview of *Mathematica* sufficient to complete the *Mathematica* modules appearing on the Web site.

Mathematica/Maple Module

Modeling Change: Springs, Driving Safety, Radioactivity, Trees, Fish, and Mammals.

Construct and interpret mathematical models, analyze and improve them, and make predictions using them.